Tiling the sphere with quadrilaterals: Cubed-sphere grids and their challenges for numerical methods

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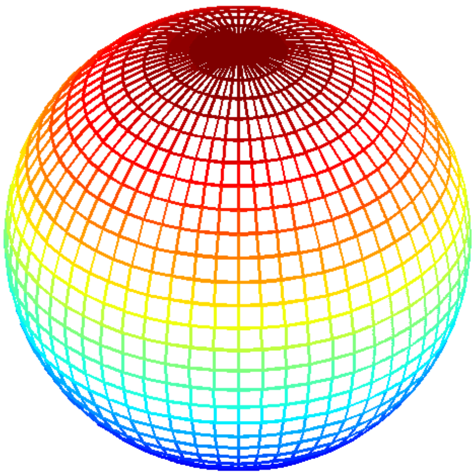
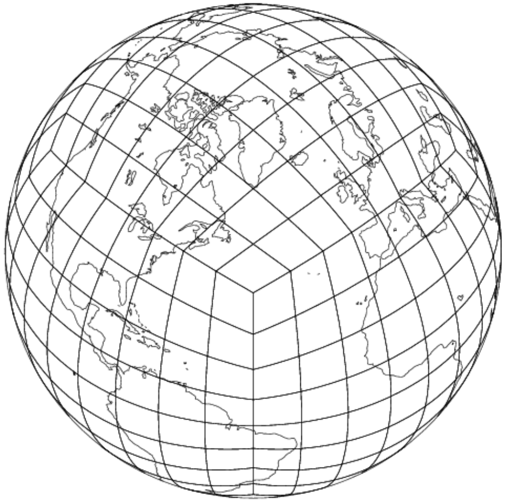
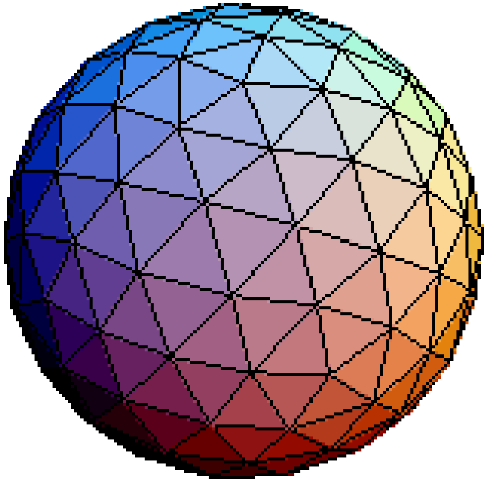
Motivation: Parallel Scalability







- Why not use latitude-longitude Grids? Well proven. Many good solutions to the "pole problem", but they degrade parallel scalability
- "petascale" ready models needed for next generation computers
- Grids which do not cluster grid points have excellent scalability:
- Cubed-sphere grids: CAM/HOMME at 25km (0.25 degree) scales to 86,400 cpus
- Icosahedral grids: Run on the JES at ~ 2km global resolution

Focus on quadrilateral grids

- A lot of numerical methods have been developed for Cartesian grids. What is the easiest way to transfer these methods to the sphere?
- Scalability requires unstructured or less-structured grids – using quads allows us to use some of the techniques developed for Cartesian grid methods.
- Dealing with spherical geometry and parallel scalability is why we are here. If compute time was not an issue, lat/lon Cartesian methods would be very difficult to beat.
- Focus on 2D discretizations of the surface of the sphere. Vertical direction can and often is treated with a completely different method

Types of quadrilateral grids

- Composite grid methods
  - Cover the sphere with patches of orthogonal Cartesian grids
  - Grids must overlap for stability
  - Interpolations between the grids make conservation quite difficult
  - Popular in generic AMR PDE solver “packages”
- Pure Quad Grids: conformal
  - Can use orthogonal Cartesian grid methods
  - Non-uniform grids - Introduces new pole-like problems
- Pure Quad Grids: non-conformal
  - Equal angle projection – very uniform grid
  - Requires a numerical method that can handle non-orthogonal unstructured grids.

The Cubed-Sphere

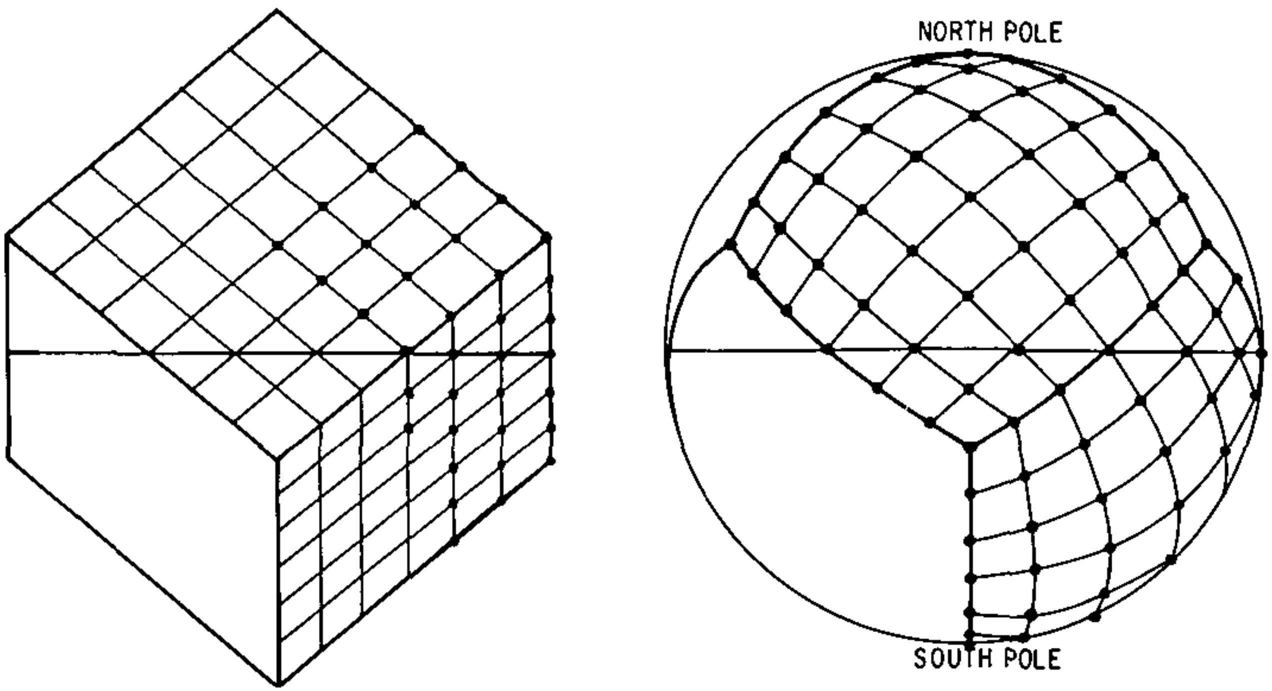


FIGURE 1.—A cubic representation of the earth. A cubic grid is shown together with the corresponding spherical grid which fits into the cubic splitting of the sphere, in the exact disposition that was used in the actual computations.

Source: Sadourny, MWR 1972

Sadourny MWR 1972

- Used the Gnomonic projection (non-orthogonal)
- Finite difference method (mass & energy conserving)
- One sided differences used at cube face boundaries
- Large truncation error at the boundary resulting in noisy solutions
- Similar approach used by Phillips MWR 1959 (two stereographic polar caps + Mercator projection tropical band) with “missing” values for FD stencils obtained by interpolation

Unfortunately, the decision to butt the coordinate systems together at a common latitude and to couple them with interpolation led to an unstable method, so the concept was abandoned.
Browning, Hack, Swarztrauber MWR 1989.

The Composite Mesh Method

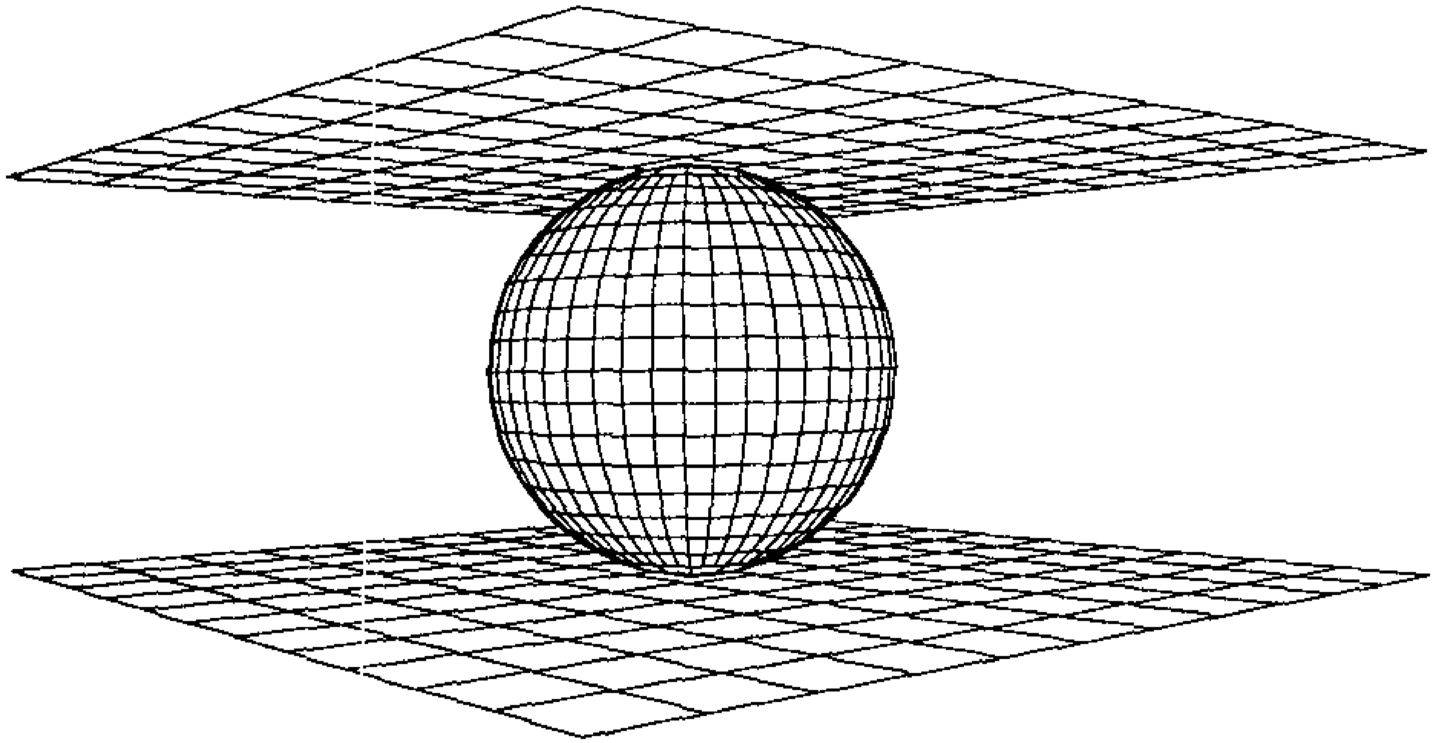


FIG. 1. The two tangent planes used in the composite-mesh method. Both planes extend slightly beyond the equator and contain a Cartesian coordinate system centered at the tangent point.

Source: Browning, Hack, Swarztrauber MWR 1989.

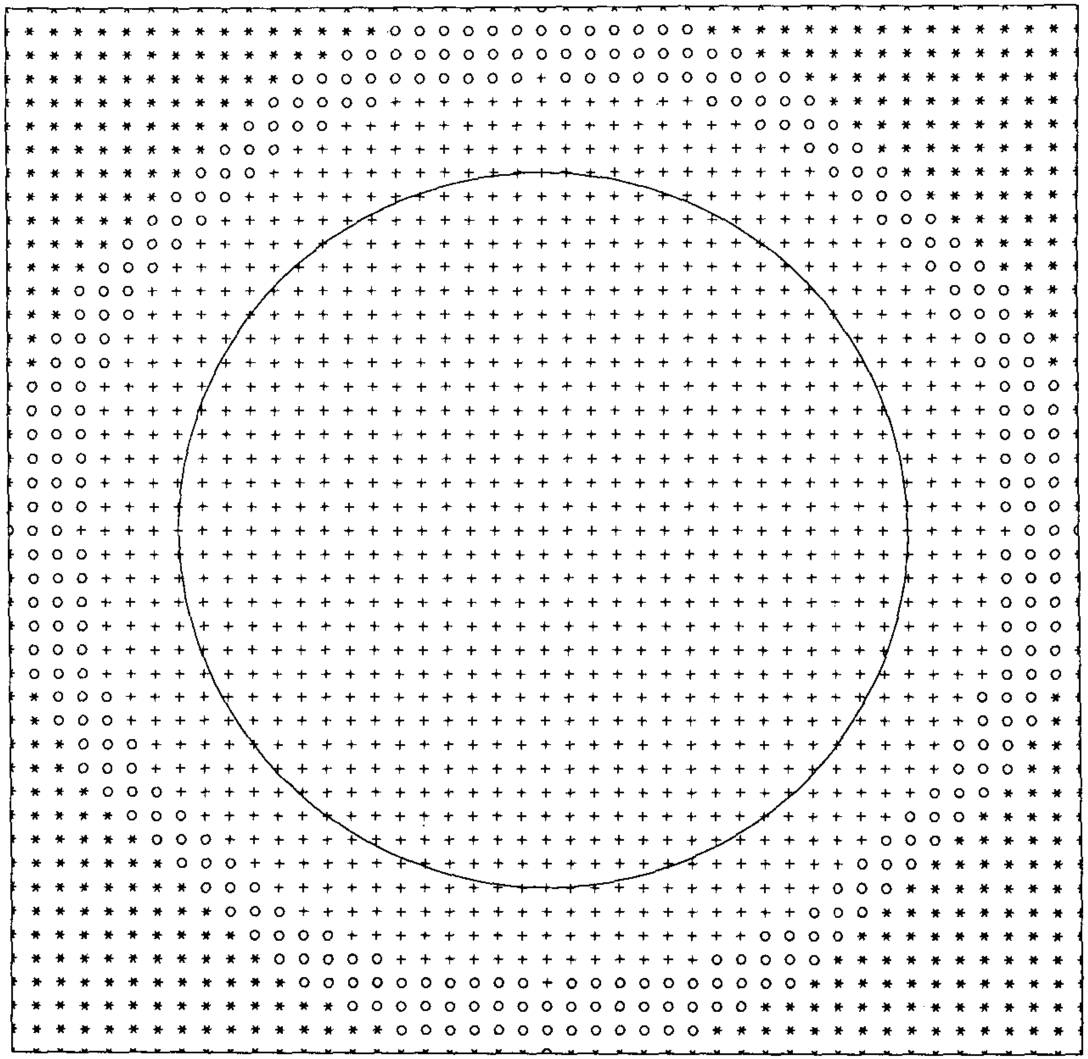


FIG. 2. The details of the composite grid mesh on either tangent plane for the case $N = 30$, $O = 7$, and $R = 3$. The circle is the image of the equator. The position of the points in D and I are marked by the symbols + and ○, respectively. The stars indicate points which are not used.

Browning, Hack, Swarztrauber MWR 1989.

The Composite Mesh Method

- Stereographic (or other orthogonal) projections used so each patch maps to a regular Cartesian grid
- Boundary points from one grid (using one coordinate system) are interior points from another grid (using a different coordinate system)
- The overlapping of all boundary points is the key to the stability of the method (Starius, Numer. Math. 1977,1980)

...there is an overlapping of the grids in the middle latitudes, and one needs to interpolate values from one grid to its neighbor in the course of the calculation. This need makes the design of a global conservative scheme impossible in practice.
Sadourny, MWR 1972

The Composite Mesh Method

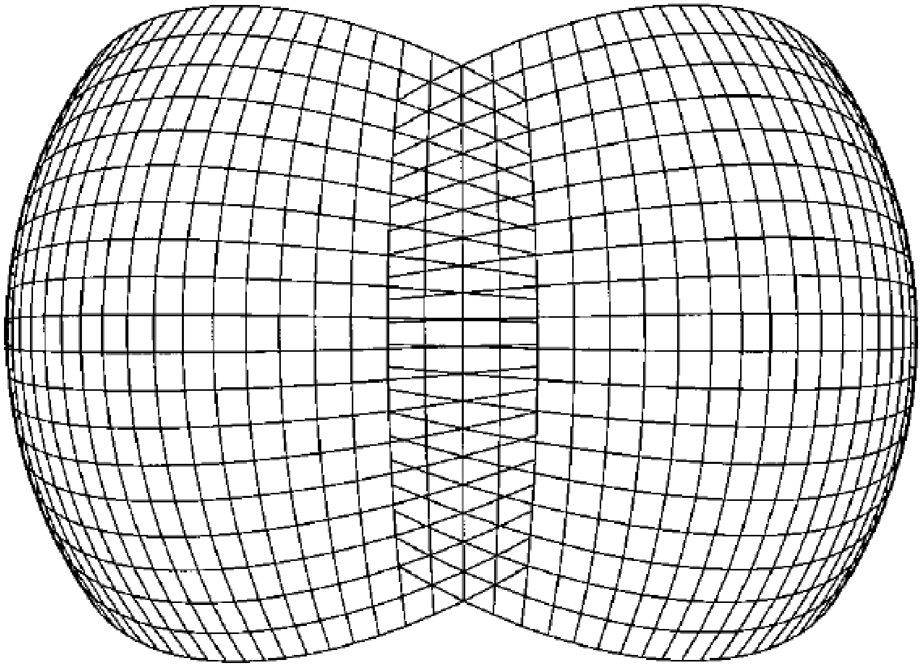


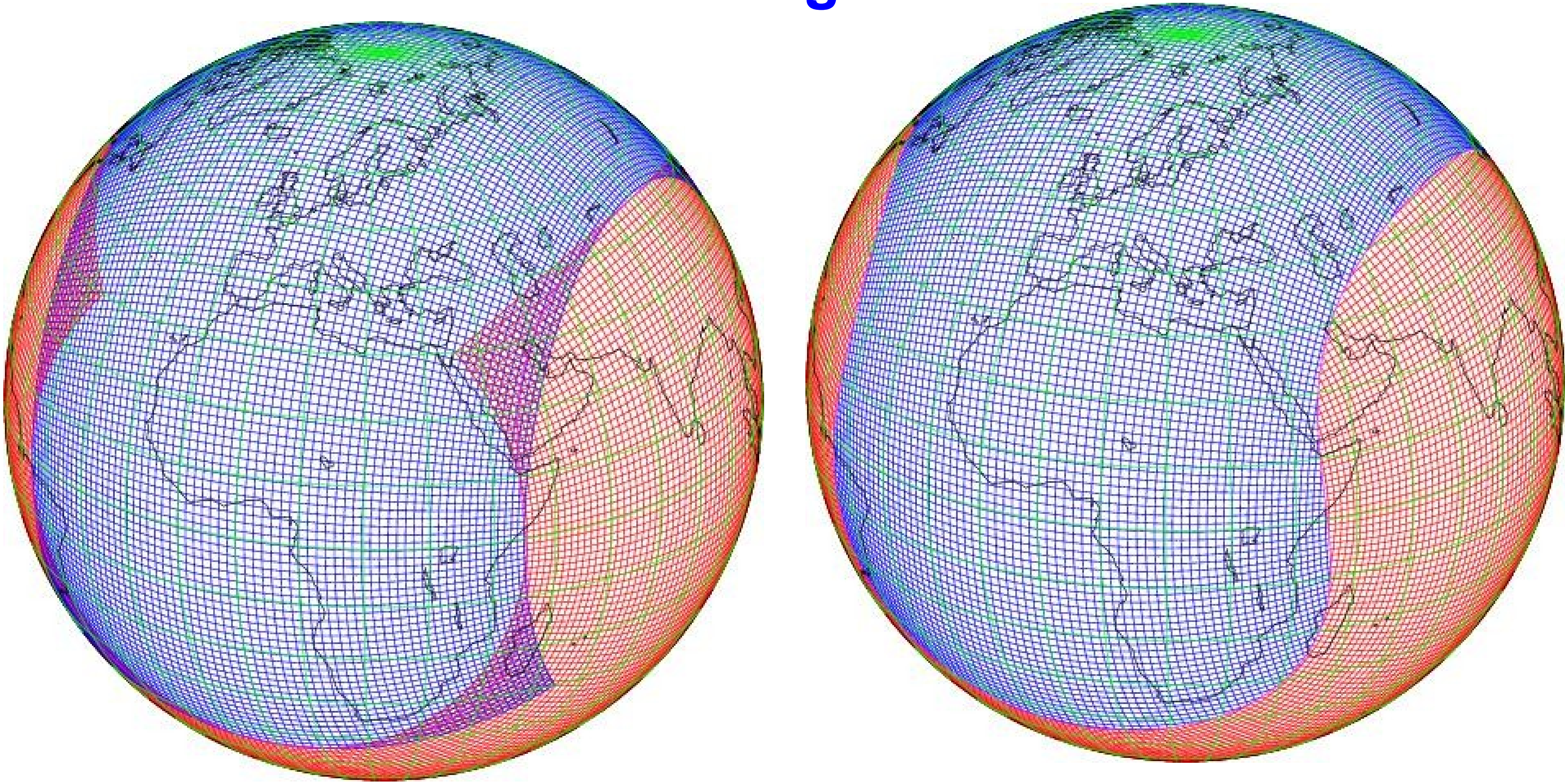
FIG. 4. View of two contiguous equatorial blocks and two of their stencils. The view is centered on the common vertical boundary line and shows the case of two grids with $N_\eta = N_\xi$ and $N_S = 2$. Notice that in this case, since both blocks use the same grid spacing, the horizontal coordinates of the stencil points of one grid coincide with the horizontal coordinates of the last two interior grid points of the contiguous block.

Ronchi, Iacono, Paulucci, JCP 1996

The Composite Mesh Method

- Ronchi, Iacono, Paulucci JCP 1996
- First use of the phrase “cubed-sphere”?
- 4th order, fully co-located A-grid like method

The Composite Mesh Method
Yin-Yang Grid



Source: R.J. Purser (NCEP) *The bi-Mercator Grid as a Global Framework for Numerical Weather Prediction*

Kageyama, Sato, Geochem. Geophs. Geosyst. 2003

Non-overlapping Quadrilateral Grids

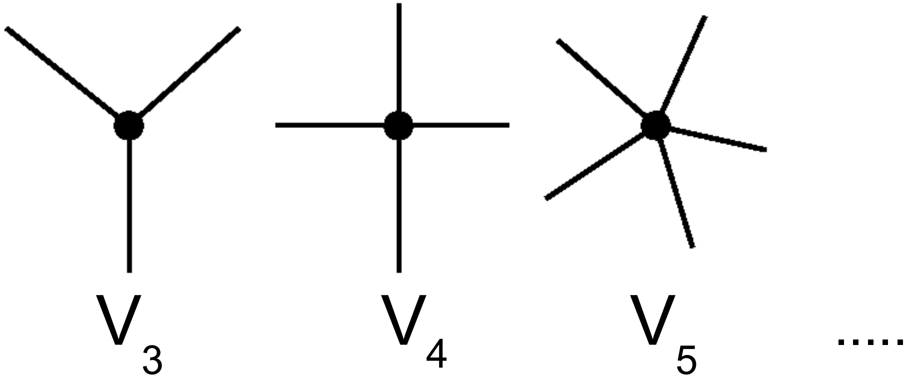
Euler's Formula for polyhedra: $V - E + F = 2$

V = number of verticies
E = number of edges
F = number of faces

Quadrilateral elements:

$E = 4F/2$

$2E = \sum j V_j$



Then: $V_3 = 8 + V_5 + 2V_6 + 3V_7 \ldots$

most uniform solution:
$V_3 = 8$, V_4=unlimited, $V_5 = V_6 = \ldots = 0$

For non-overlapping quadrilateral grids,
The cubed sphere is the only reasonable
choice!

Conformal Cubed-Sphere Grids

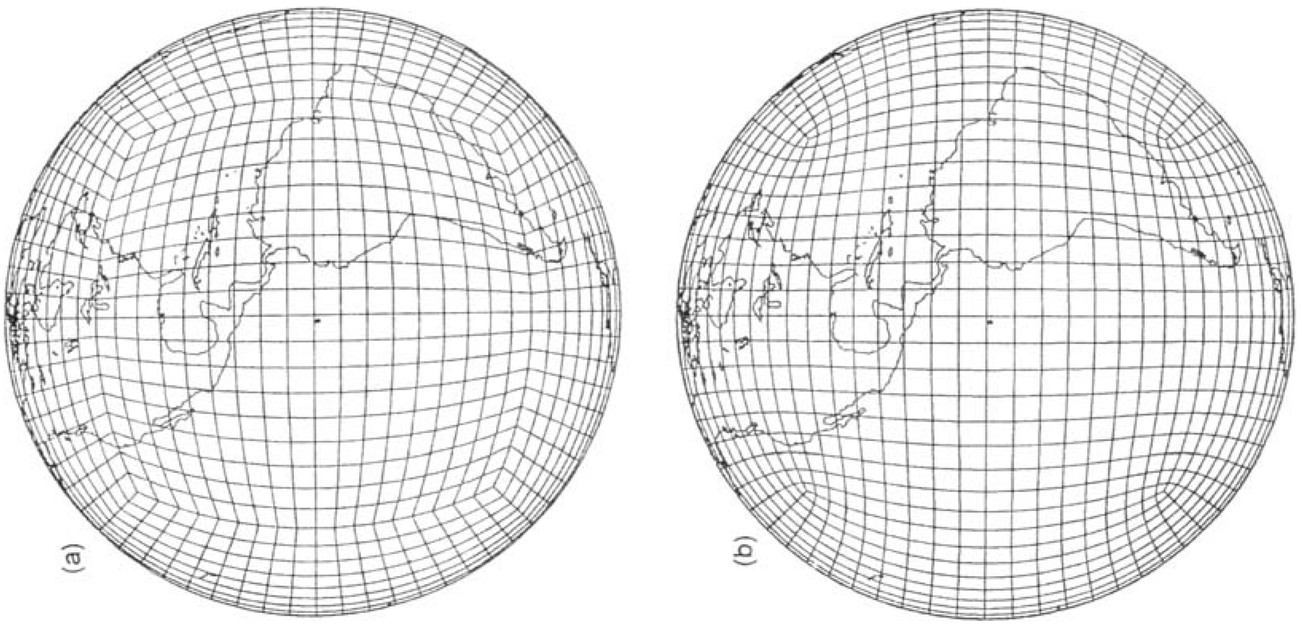


Figure 1. (a) Spherical-cubic grid using the gnomonic projection. Adjacent grid lines subtend equal angles at the centre of the sphere. (b) Spherical-cubic grid obtained by conformal transformation to map domain consisting of the square faces of the unit cube. Grid lines are equally spaced in the map coordinates.

Rancic, Purser, Mesinger QJRMS 1996
McGregor Atmos. Ocean 1996

Conformal Cubed-Sphere Grids

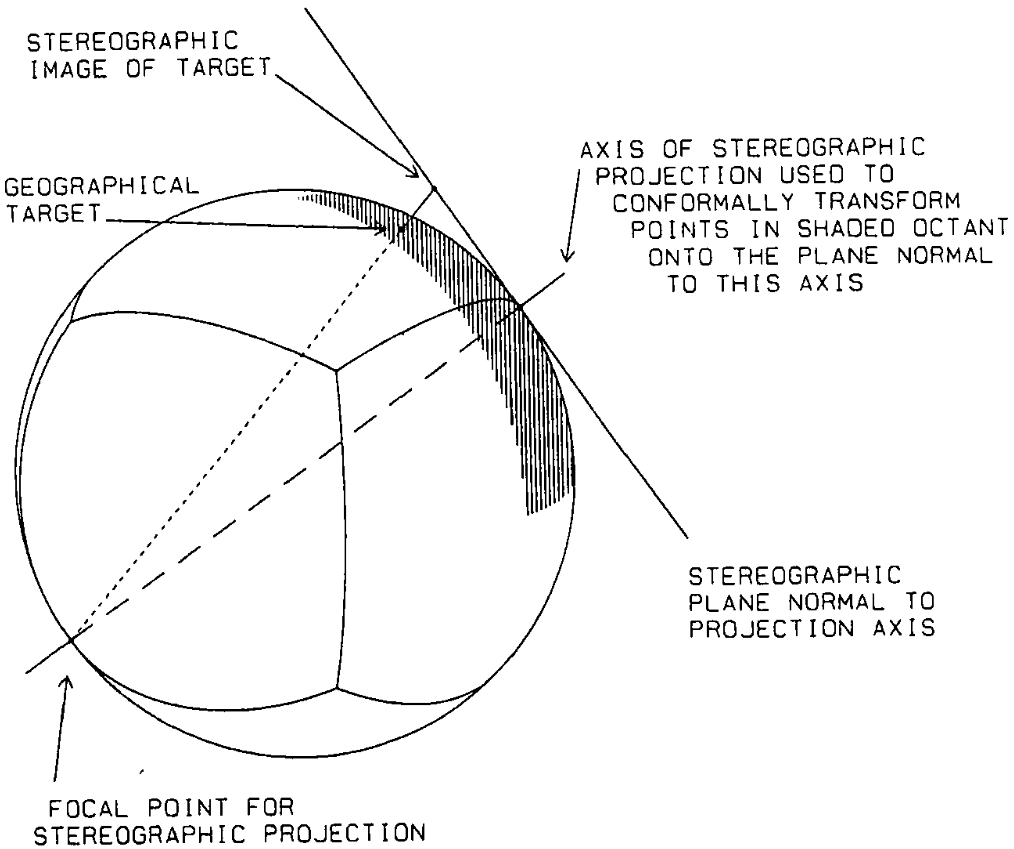


Figure A.1. Schematic depiction of the construction of the image on a plane of a target on the sphere. The image plane for any target is parallel to the sphere's surface at the point corresponding to the nearest corner of the inscribed cube. Therefore, the points on the sphere that share a common stereographic projection comprise an octant (shaded region).

Rancic, Purser, Mesinger QJRMS 1996

Conformal Cubed-Sphere Grids

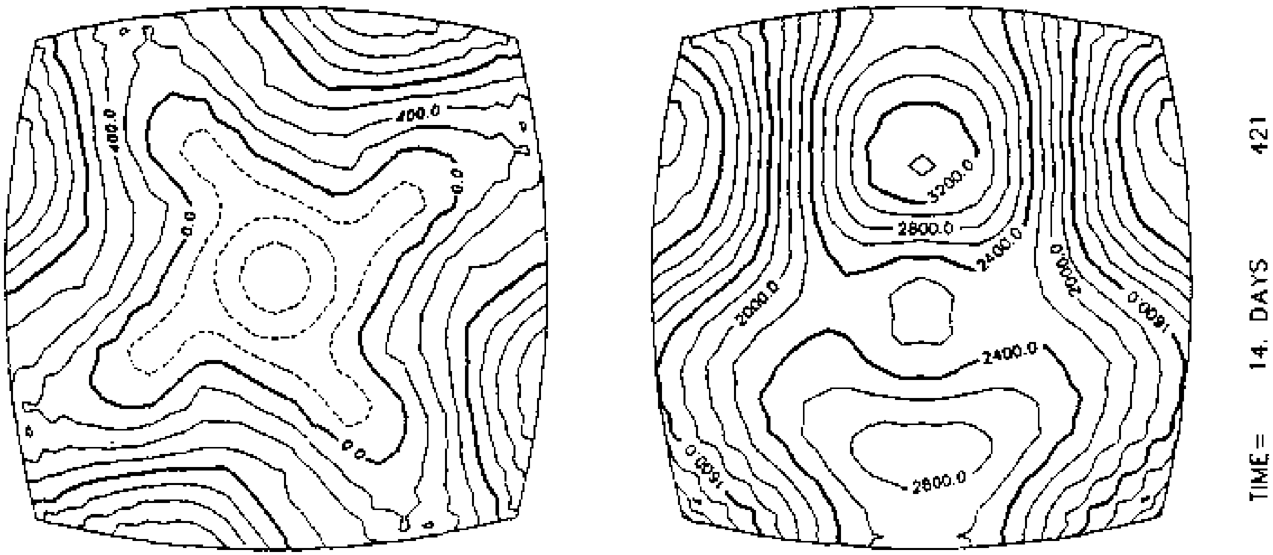


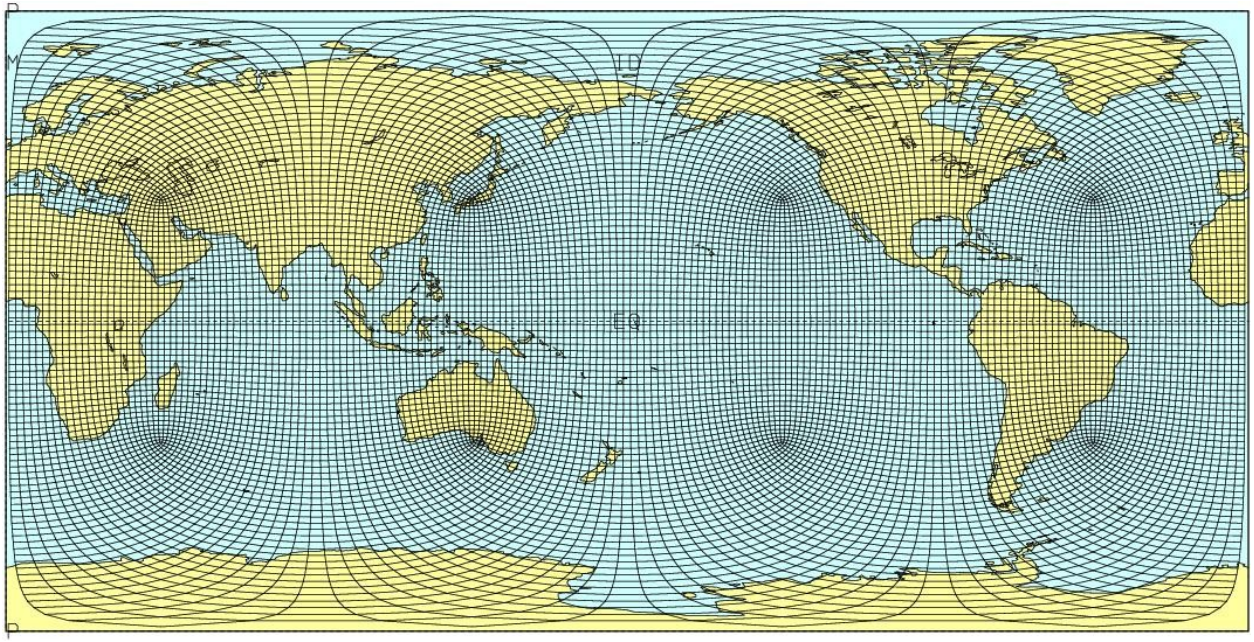
Figure 5. Numerical solution after 14 days of integration with the Rossby–Haurwitz zonal wave-4, using t
E-grid, with the resolution of 421 scalar gridpoints on each of the faces of the spherical cube. The upper face (w
the north pole in the middle) and one of the lateral faces are shown in the stereographic projection.

Sample cubed-sphere output from
Rancic, Purser, Mesinger QJRMS 1996

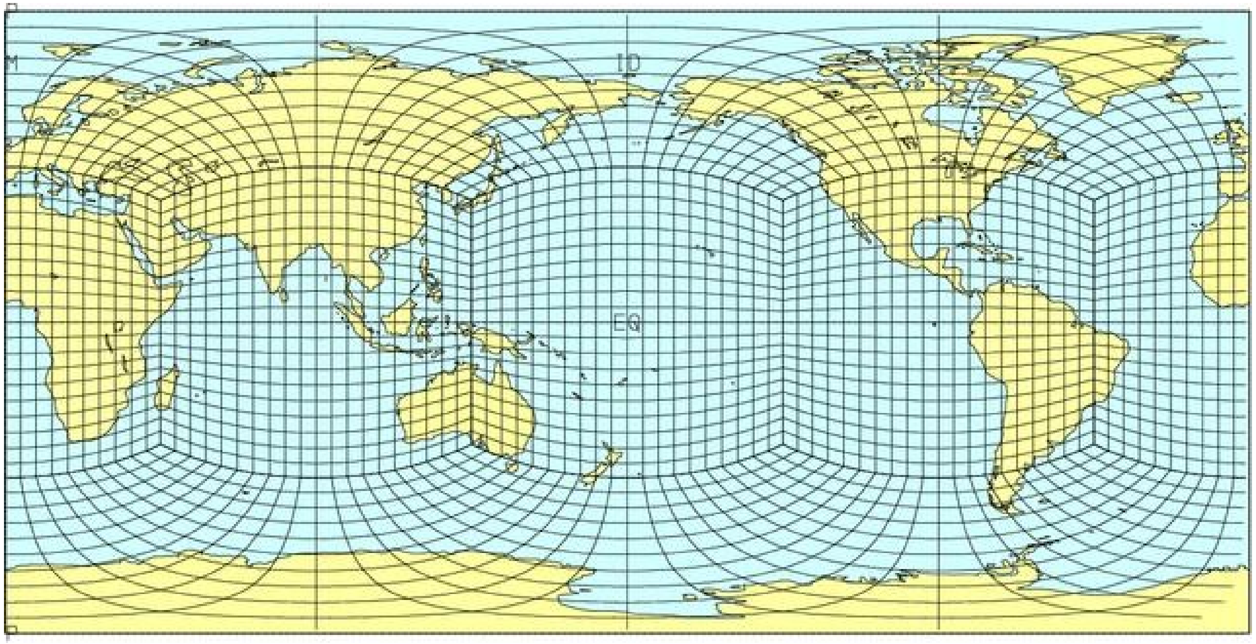
Conformal Cubed-Sphere Grids

- Orthogonal grid – can use your favorite Cartesian grid method (but need to be careful at the 8 corner points)
- Used by several modeling groups including two that are here at this seminar (MIT and GEF)
- But at higher resolutions you are faced by a pole-like CFL problem caused by the clustering of the grid at the 8 corner points

Cubed-sphere Projections



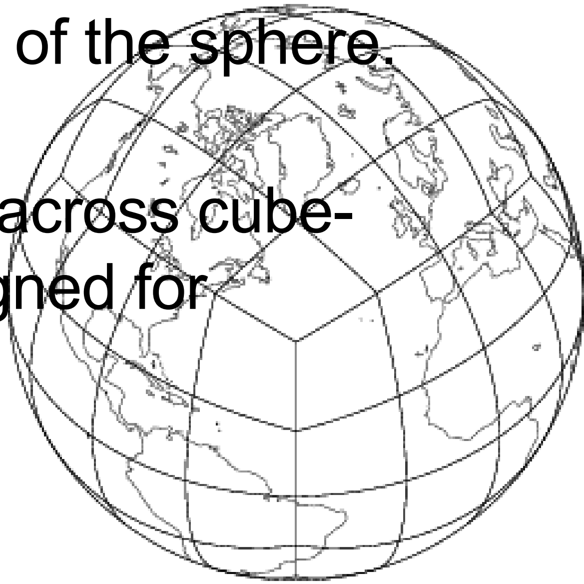
Conformal
8 "poles"



Gnomonic:
Non-orthgonal coordinate system

Source: J. McGregor (CSIRO) *Some features of the dynamical formulation of CCAM,* PDEs on the sphere, 2006

Equal Angle Cubed-Sphere Grids



- Gnomonic projection: project from the center of the sphere. Non-orthogonal coordinate system
- Coordinate lines do not extend continuously across cube-faces. Have to use numerical methods designed for unstructured grids.
- Directionally split techniques no longer work
- However:
  - Equal angle spacing for quasi-uniform resolution. Coordinate lines are arcs of great circles
  - Used by two groups at this seminar (CAM/HOMME and NASA FVcubed)

Taylor, Tribbia, Iskandarani, JCP 1997

Numerical Methods for non-orthogonal unstructured grids

- Flux based: finite volume, discontinuous Galerkin
- Continuous Galerkin (finite element, spectral finite element)
- Mimetic
- Others?

Flux based Methods: Mass Conservation

Start with advection equation in conservation form

$$\frac{\partial h}{\partial t}+\nabla\cdot(h\boldsymbol{u})=0$$

Integrate over a small control volume:

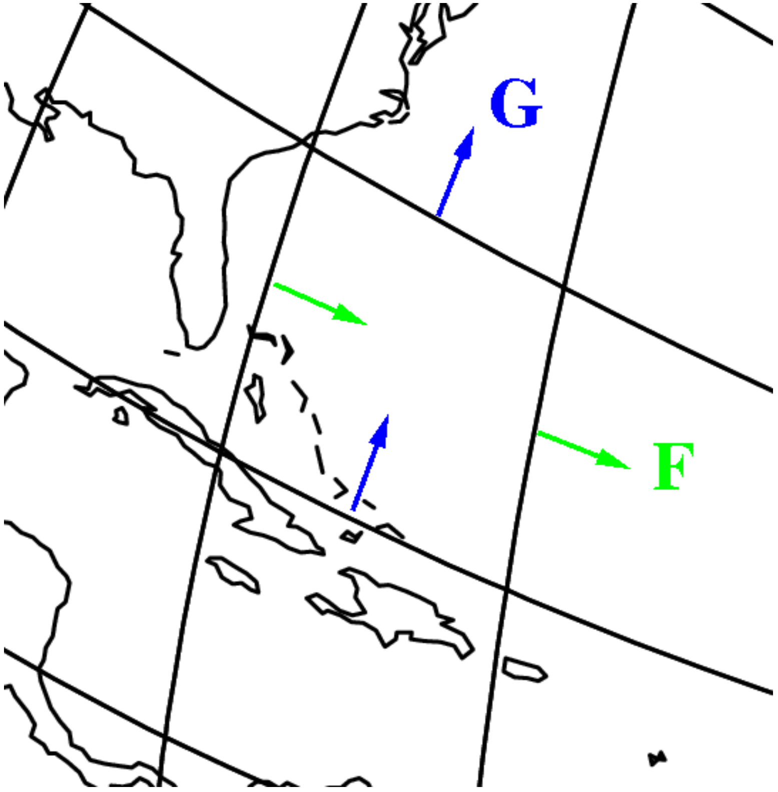
$$\int_{\Omega}\frac{\partial h}{\partial t}+\int_{\Omega}\nabla\cdot(h\boldsymbol{u})=0$$

Apply divergence theorem:

$$\frac{\partial \bar{h}}{\partial t}+\oint h\boldsymbol{u}\cdot\hat{n}=0$$

Approximate Flux on cell edges:

$$\bar{h}_{t+1}=\bar{h}_t+\boldsymbol{F}+\boldsymbol{G}$$



Flux based methods: Energy Conservation?

- FV methods on quad grids: typically can not conserve energy in a climate model: would need to advect total energy instead of temperature
- Why not advect total energy?

Conservation in Finite Element Methods

- Finite element method solves an integral form of the equations
- Approximates the function space that the solution lives in – then computes derivatives exactly (as opposed to approximating the derivative operators)
- Finite element methods can be made *compatible*
- *Compatible* methods can conserve both mass and energy

Example from the ASP seminar of compatible methods:
CAM/HOMME and CAM/EUL

Shallow Water Equations
2D Flow on the surface of the sphere

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{\omega} + f)\hat{\boldsymbol{k}} \times \boldsymbol{u} + \nabla(\frac{1}{2}\boldsymbol{u}^2 + gH) = 0$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\boldsymbol{u}) = 0$$

$\boldsymbol{u}$ = velocity field
$\boldsymbol{\omega}$ = vorticity
h = atmosphere thickness
H = atmospheric height $h + h_s$

Shallow Water Equations
Weak Formulation

$$\psi\,,\vec{\psi} \in H_1 \qquad \boldsymbol{u}\,, h \in H_1$$

Solve system of scalar integral equations for all test functions

$$\psi\,,\vec{\psi} \in H_1 \qquad \boldsymbol{u}\,, h \in H_1$$

With the usual area weighted integral over the surface of the sphere:

$$\int () = \iint () \, r \cos(\theta) \, d\theta \, d\lambda$$

Spectral Finite Element Discretization

- Tile the sphere with elements
- H_1^d = set of all C^0 functions which are polynomials up to degree *d* within the elements.
- Construct test/basis functions for H_1^d which have compact support over few elements

$$\mathrm{span}\{\phi_i\} = H_1^d$$

- Solve finite set of scalar integral equations exactly

Compatible Numerical Methods

Discrete operators and discrete integral satisfy continuum properties:

$$\int \nabla \cdot (p\boldsymbol{v}) = \int p \nabla \cdot \boldsymbol{v} + \int \boldsymbol{v} \cdot \nabla p = 0$$

$$\int \nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \int \boldsymbol{v} \cdot \nabla \times \boldsymbol{u} - \int \boldsymbol{u} \cdot \nabla \times \boldsymbol{v} = 0$$

$$\nabla \times \nabla p = 0$$

$$\nabla \cdot \nabla \times \boldsymbol{u} = 0$$

- Integration by parts insures conservation
- Curl Grad = 0 can improve vorticity evolution
- Many schemes have this property on orthogonal Cartesian grids
- Difficult to preserve on unstructured grids
- Spectral Element Method is Compatible on very general unstructured grids.

Global Conservation: Mass

Integrate advection equation over the entire sphere:

$$\int \frac{\partial h}{\partial t} + \int \nabla \cdot (h\,\boldsymbol{u}) = 0$$

Apply divergence theorem: $$\frac{d}{dt} \int h = 0$$

A numerical scheme will conserve global mass if its discrete approximation to the integral and divergence operator used for advection satisfy:

$$\int \nabla \cdot (\boldsymbol{v}) = 0$$

Global Conservation: Tracer Mass

Advection equation in non-conservation form:

$$\frac{\partial q}{\partial t} + \boldsymbol{u} \cdot \nabla q = 0$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (h \boldsymbol{u}) = 0$$

$$Mass = \int h q$$

Multiply and Integrate over the sphere:

$$\int h \frac{\partial q}{\partial t} + \int h \boldsymbol{u} \cdot \nabla q = 0$$

$$\int q \frac{\partial h}{\partial t} + \int q \nabla \cdot (h \boldsymbol{u}) = 0$$

Global Conservation: Tracer Mass

Sum:

$$\frac{d}{dt}\int q\,h+\int h\,\boldsymbol{u}\cdot\nabla q+\int q\,\nabla\cdot(h\,\boldsymbol{u})=0$$

A numerical scheme will conserve global tracer mass if the discrete div and grad operators used for advection are adjoints in the inner product defined by the discrete approximation to the integral. (integration by parts)

$$\int p\,\nabla\cdot\boldsymbol{v}=-\int \boldsymbol{v}\cdot\nabla p$$

By taking p=1, we also have: $\int \nabla\cdot(\boldsymbol{v})=0$

Energy Conservation

- Dry Primitive Equations
- Sigma coordinates in the vertical
- Show conservation in the unforced, inviscid equations

$$M = \int p_s \; dA$$

$$KE = \frac{1}{2}\int_0^1 \int p_s \boldsymbol{u}\cdot\boldsymbol{u} \; dA\, d\sigma$$

$$IE = C_p \int_0^1 \int p_s T \; dA\, d\sigma$$

Surface integral over the unit sphere:

$$\int (\,)\, dA = \iint (\,) \cos(\theta)\, d\theta\, d\lambda$$

Energy Conservation: KE

$$\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{\omega}+f)\hat{\boldsymbol{k}}\times\boldsymbol{u}+\nabla\left(\frac{1}{2}\boldsymbol{u}^2+\Phi\right)+\frac{RT}{p_s}\nabla p_s+\dot{\sigma}\frac{\partial \boldsymbol{u}}{\partial \sigma}=0$$

$$\frac{\partial p_s}{\partial t}+\nabla\cdot(p_s\boldsymbol{u})+\frac{\partial}{\partial \sigma}(p_s\dot{\sigma})=0$$

Multiply and Integrate over the sphere:

$$\frac{d}{dt}KE=\iint RT\,\boldsymbol{u}\cdot\nabla p_s+\iint p_s\boldsymbol{u}\cdot\nabla\Phi$$

$$=\iint RT\,\boldsymbol{u}\cdot\nabla p_s+\iint \nabla\cdot(p_s\boldsymbol{u})\int_{\sigma}^{1}\frac{RT}{\sigma}\,d\sigma$$

Energy Conservation: IE

$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T + \dot{\sigma} \frac{\partial T}{\partial \sigma} - \frac{RT}{C_p} \frac{\omega}{p} = 0$$

$$\frac{\partial p_s}{\partial t} + \nabla \cdot (p_s \boldsymbol{u}) + \frac{\partial}{\partial \sigma} (p_s \dot{\sigma}) = 0$$

Multiply and Integrate over the sphere:

$$\frac{d}{dt} IE = \iint RT\, p_s \frac{\omega}{p}$$

$$= \iint RT\, \boldsymbol{u} \cdot \nabla p_s - \iint \frac{RT}{\sigma} \int_0^{\sigma} \nabla \cdot (p_s \boldsymbol{u})\, d\sigma$$

Total Energy Conservation

Such a numerical method then satisfies a discrete version of:

$$\frac{d}{dt}KE = -\iint RT\,\boldsymbol{u}\cdot\nabla p_s + \iint \nabla\cdot(p_s\boldsymbol{u})\int_\sigma^1 \frac{RT}{\sigma}\,d\sigma$$

$$\frac{d}{dt}IE = \iint RT\,\boldsymbol{u}\cdot\nabla p_s - \iint \frac{RT}{\sigma}\int_0^\sigma \nabla\cdot(p_s\boldsymbol{u})\,d\sigma$$

- Notes:
  - Momentum advection exactly preserves KE
  - Temperature advection exactly preserves IE
  - Mass advection exactly preserves mass
  - KE <-> PE transfer terms exactly balance

Total Energy Conservation: Requirements

- Conservative vertical coordinate system (Simmons and Burridge, or Lagrangian)
  - Carefully constructed hydrostatic equation and matching equations for dσ/dt
  - Vertical derivative operator can be integrated by parts
- Horizontal discretization:
  - div/grad operators used in equations can be integrated by parts

$$\int p \nabla \cdot \boldsymbol{v} \quad dA = -\int \boldsymbol{v} \cdot \nabla p \quad dA$$

$$\int_0^1 p \frac{d\,q}{d\,\sigma} d\,\sigma = -\int_0^1 q \frac{d\,p}{d\,\sigma} d\,\sigma \quad + \quad b.c.\,terms$$

SUMMARY
Quadrilateral Grids on the Sphere:

- Composite grid methods
  - Use your favorite orthogonal Cartesian grid methods
  - Interpolations make it difficult to maintain conservation
- Pure Quad Grids: conformal
  - Can use orthogonal Cartesian grid methods (some care needed at 8 corner points)
  - Non-uniform grids - Introduces new pole-like problem
- Pure Quad Grids: non-conformal
  - Equal angle projection – very uniform grid
  - Requires a numerical methods designed for non-orthogonal unstructured grids.