SPECTRAL TRANSFORM METHOD: EULERIAN AND SEMI-LAGRANGIAN

David Williamson
National Center for Atmospheric Research

SPECTRAL METHOD (1 DIMENSION)

$$q(x_j, t) = \sum_{k=-K}^{K} q_k(t) e^{ikx_j}$$

$q_{-k} = q_k^*$

$q(x_j, t)$ and $q_k(t)$ equivalent if $K = J/2$

$$\frac{1}{J} \sum_{j=1}^{J} e^{ikx_j} e^{-ilx_j} = \delta_{kl}$$

$$\frac{1}{J} \sum_{j=1}^{J} q(x_j, t) e^{-ikx_j} = q_k(t)$$

ADVECTION, CONSTANT U

$$\frac{\partial q}{\partial t} + U\frac{\partial q}{\partial x} = 0$$

$$q(x_j, t) = \sum_{k=-K}^{K} q_k(t) e^{ikx_j}$$

$$\frac{1}{2\Delta t}\sum_{k=-K}^{K}(q_k^{n+1} - q_k^{n-1})e^{ikx_j} + U\sum_{k=-K}^{K} ikq_k^n e^{ikx_j} = 0$$

Multiply by e^{-ikx_j} and sum over the x_j from 1 to J

$$\frac{1}{2\Delta t}(q_k^{n+1} - q_k^{n-1}) + Uikq_k^n = 0$$

STABILITY CONDITION

$$\frac{1}{2\Delta t}(q_k^{n+1} - q_k^{n-1}) + Uikq_k^n = 0$$

Assume $q_k^n = e^{i\nu_k n\Delta t}$

$$\frac{i \sin \nu_k \Delta t}{\Delta t} + ikU = 0 \quad , \qquad \nu_k = -\frac{1}{\Delta t} \arcsin(kU\Delta t)$$

$$|kU\Delta t| \leq 1$$

$k \leq J/2$, $\Delta x = 2\pi/J$

$$\left| U \frac{\Delta t}{\Delta x} \right| \leq \frac{1}{\pi}$$

ADVECTION, VARIABLE U

$$\frac{\partial q}{\partial t} + U\frac{\partial q}{\partial x} = 0$$

$$\frac{1}{\Delta t}\sum_{k=-K}^{K}(q_k^{n+1} - q_k^{n-1})e^{ikx_j} + \sum_{l=-K}^{K} U_l^n e^{ilx_j} \sum_{m=-K}^{K} imq_m^n e^{imx_j} = 0$$

Multiply by e^{-ikx_j} and sum over the x_j from 1 to J

$$\frac{1}{\Delta t}(q_k^{n+1} - q_k^{n-1}) + \sum_{l+m=k}^{K} U_l^n imq_m^n = 0$$

SPECTRAL TRANSFORM METHOD

$$U(x_j, n\Delta t) = \sum_{k=-K}^{K} U_k^n e^{ikx_j}$$

$$\frac{\partial q(x_j, n\Delta t)}{\partial x} = \sum_{k=-K}^{K} ikq_k^n e^{ikx_j}$$

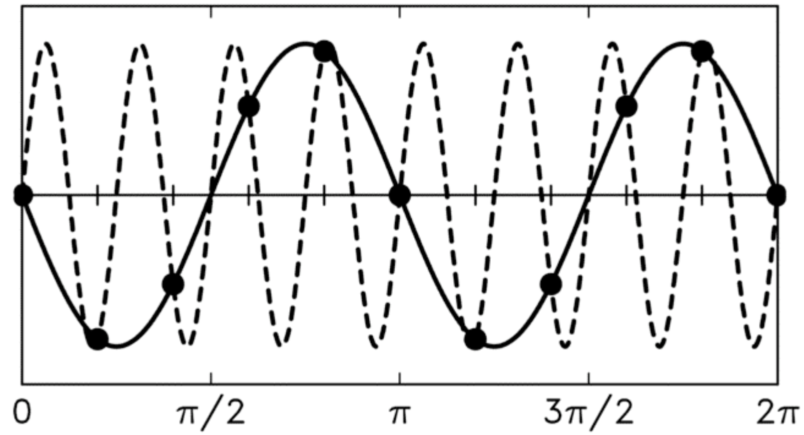
$U(x_j)\frac{\partial q}{\partial x}(x_j)$ calculated on the grid

$$\frac{\partial q_k^n}{\partial t} = -\left(U\frac{\partial q}{\partial x}\right)_k^n = -\frac{1}{J}\sum_{j=1}^{J}\left[U(x_j, n\Delta t)\frac{\partial q(x_j, n\Delta t)}{\partial x}\right]e^{-ikx_j}$$

$K \leq J/2$

$$\frac{1}{2\Delta t}(q_k^{n+1} - q_k^{n-1}) + \frac{1}{J}\sum_{j=1}^{J}\left[U(x_j, n\Delta t)\frac{\partial q(x_j, n\Delta t)}{\partial x}\right]e^{-ikx_j} = 0$$

ALIASING



$$e^{ilx_j}e^{imx_j} = e^{i(l+m)x_j}$$

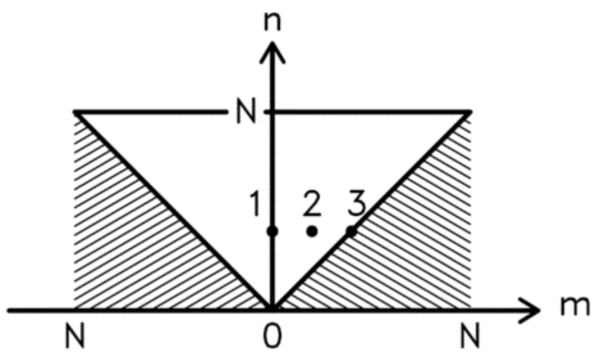
$$\sum_{l=-K}^{K} U_l e^{ilx_j} \sum_{m=-K}^{K} imq_m e^{imx_j} = \sum_{l=-K}^{K}\sum_{m=-K}^{K} imq_m U_l e^{i(l+m)x_j}$$

if $(l+m) > J/2$, then appears as $k < J/2$

choose K so aliased waves fall in range$(K, J/2)$

instead of $K = J/2$, $K \leq (J-1)/3$

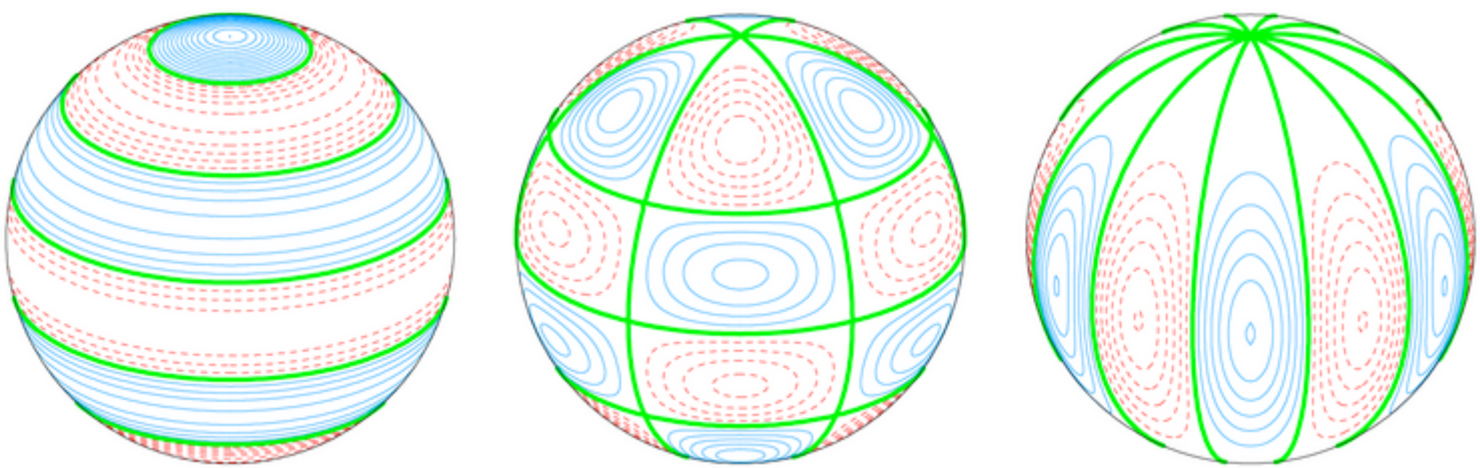
SPHERICAL HARMONICS

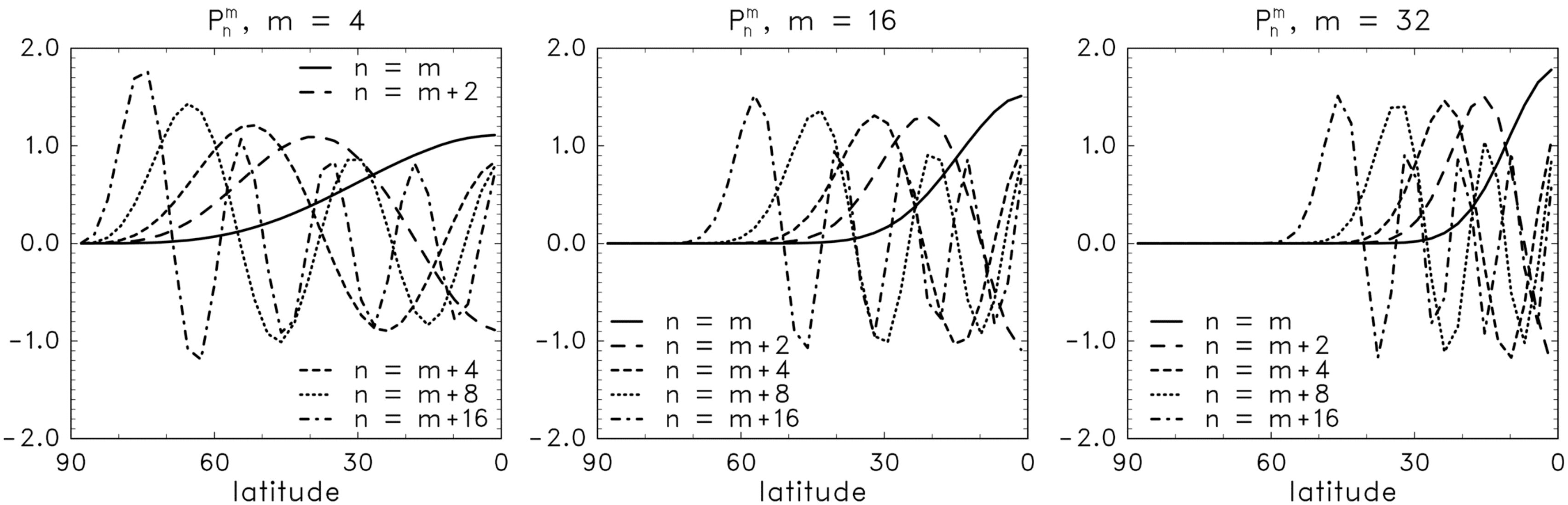


$$q(\lambda, \varphi, t) = \sum_{n=0}^{N} \sum_{m=-n}^{n} q_n^m(t) Y_n^m(\lambda, \mu)$$

$$Y_n^m(\lambda, \mu) = P_n^m(\mu) e^{im\lambda}$$

$$\mu = \sin \varphi$$





P_n^m, m = 4
P_n^m, m = 16
P_n^m, m = 32
n = m
n = m+2
n = m+4
n = m+8
n = m+16
2.0
1.0
0.0
−1.0
−2.0
90
60
30
0
latitude

$$q(\lambda, \varphi, t) = \sum_{n=0}^{N} \sum_{m=-n}^{n} q_n^m(t) Y_n^m(\lambda, \mu)$$

$$q_n^m = \int_{-1}^{1} \frac{1}{2\pi} \int_0^{2\pi} q(\lambda, \mu) e^{-im\lambda} d\lambda P_n^m(\mu) d\mu$$

$$q^m(\mu) = \frac{1}{2\pi} \int_0^{2\pi} q(\lambda, \mu) e^{-im\lambda} d\lambda$$

$$q_n^m = \sum_{j=1}^{J} q^m(\mu_j) P_n^m(\mu_j) w_j$$

$$\mu_j : \ J \text{ roots of } P_J(\mu) \ , \quad w_j = \frac{2(1-\mu_j^2)}{[J \ P_{J-1}(\mu_j)]^2} \ , \quad \sum_{j=1}^{J} w_j = 2.0$$

PROPERTIES OF SPHERICAL HARMONICS

$$Y_n^m(\lambda,\mu) = P_n^m(\mu)e^{im\lambda}$$

$$\nabla q = \frac{1}{a}\left(\frac{1}{\cos\varphi}\frac{\partial q}{\partial\lambda}\hat{\mathbf{i}}\ ,\frac{\partial q}{\partial\varphi}\hat{\mathbf{j}}\right)$$

$$\frac{\partial Y_n^m}{\partial\lambda} = imY_n^m$$

$$\cos\varphi\frac{\partial Y_n^m}{\partial\varphi} = (n+1)\epsilon_n^m Y_{n-1}^m - n\epsilon_{n+1}^m Y_{n+1}^m \quad , \qquad \epsilon_n^m = \left(\frac{n^2-m^2}{4n^4-1}\right)^{\frac{1}{2}}$$

$$\nabla^2 q = \frac{1}{a^2}\left[\frac{1}{\cos^2\varphi}\frac{\partial^2 q}{\partial\lambda^2} + \frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial q}{\partial\varphi}\right)\right]$$

$$\nabla^2 Y_n^m = \frac{-n(n+1)}{a^2}Y_n^m$$

SHALLOW WATER EXAMPLE

$$\zeta = \frac{1}{a\cos\varphi}\left[\frac{\partial v}{\partial \lambda} - \frac{\partial}{\partial \varphi}(u\cos\varphi)\right] \quad , \quad \delta = \frac{1}{a\cos\varphi}\left[\frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \varphi}(v\cos\varphi)\right]$$

$$U = u\cos\varphi \quad , \quad V = v\cos\varphi$$

$$\frac{\partial \zeta}{\partial t} = -\frac{1}{a\cos^2\varphi}\frac{\partial}{\partial \lambda}\left[(\zeta + f)\,U\right] - \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left[(\zeta + f)\,V\right]$$

$$\frac{\partial \delta}{\partial t} = \frac{1}{a\cos^2\varphi}\frac{\partial}{\partial \lambda}\left[(\zeta + f)\,V\right] - \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left[(\zeta + f)\,U\right] - \nabla^2\left[gh + \frac{U^2 + V^2}{2\cos^2\varphi}\right]$$

$$\frac{\partial h}{\partial t} = -\frac{1}{a\cos^2\varphi}\frac{\partial}{\partial \lambda}(hU) - \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}(hV)$$

EXPLICIT APPROXIMATIONS $\qquad \zeta_{i,j}^{n} \rightarrow \zeta^{n}$

$$\frac{\partial \zeta}{\partial t} = -\frac{1}{a\cos^2\varphi}\frac{\partial}{\partial \lambda}\left[(\zeta + f)\,U\right] - \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left[(\zeta + f)\,V\right]$$

$$\zeta^{n+1} = \zeta^{n-1} - \frac{2\Delta t}{a\cos^2\varphi}\frac{\partial}{\partial \lambda}\left[(\zeta^n + f)\,U^n\right] - \frac{2\Delta t}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left[(\zeta^n + f)\,V^n\right]$$

$$\frac{\partial \delta}{\partial t} = \frac{1}{a\cos^2\varphi}\frac{\partial}{\partial \lambda}\left[(\zeta + f)\,V\right] - \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left[(\zeta + f)\,U\right]$$
$$-\nabla^2\left[gh + \frac{U^2 + V^2}{2\cos^2\varphi}\right]$$

$$\delta^{n+1} = \delta^{n-1} + \frac{2\Delta t}{a\cos^2\varphi}\frac{\partial}{\partial \lambda}\left[(\zeta^n + f)\,V^n\right] - \frac{2\Delta t}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left[(\zeta^n + f)\,U^n\right]$$
$$-2\Delta t\nabla^2\left[gh^n + \frac{(U^n)^2 + (V^n)^2}{2\cos^2\varphi}\right]$$

$$\frac{\partial h}{\partial t} = -\frac{1}{a\cos^2\varphi}\frac{\partial}{\partial\lambda}(hU) - \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(hV)$$

$$h^{n+1} = h^{n-1} - \frac{2\Delta t}{a\cos^2\varphi}\frac{\partial}{\partial\lambda}(h^n U^n) - \frac{2\Delta t}{a\cos\varphi}\frac{\partial}{\partial\varphi}(h^n V^n)$$

$$
\begin{aligned}
\left\{\frac{\partial}{\partial\lambda}\left[(\zeta+f)V\right]\right\}^m =& \frac{1}{2\pi}\int_o^{2\pi}\frac{\partial\left[(\zeta+f)V\right]}{\partial\lambda}e^{-im\lambda}d\lambda \\
=& -\frac{1}{2\pi}\int_o^{2\pi}\left[(\zeta+f)V\right]\frac{\partial\left(e^{-im\lambda}\right)}{\partial\lambda}d\lambda \\
=& im\frac{1}{2\pi}\int_o^{2\pi}\left[(\zeta+f)V\right]e^{-im\lambda}d\lambda
\end{aligned}
$$

$$
\left\{\frac{\partial}{\partial\lambda}\left[(\zeta+f)V\right]\right\}_n^m = im\sum_{j=1}^{J}\left[(\zeta+f)V\right]_j^m P_n^m(\mu_j)w_j
$$

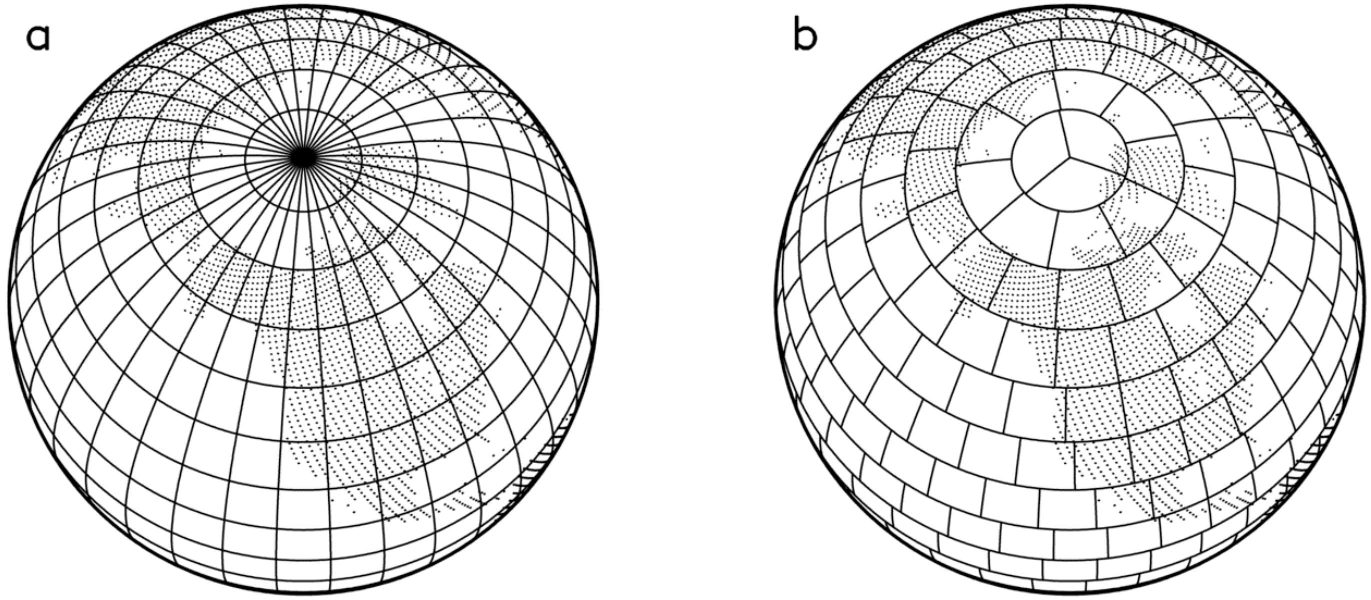
$$\left\{\frac{\partial}{\partial\mu}\left[(\zeta+f)U\right]\right\}_n^m = \int_{-1}^{1}\frac{\partial}{\partial\mu}\left[(\zeta+f)U\right]^m P_n^m d\mu$$
$$= -\int_{-1}^{1}\left[(\zeta+f)U\right]^m \frac{dP_n^m}{d\mu}d\mu$$

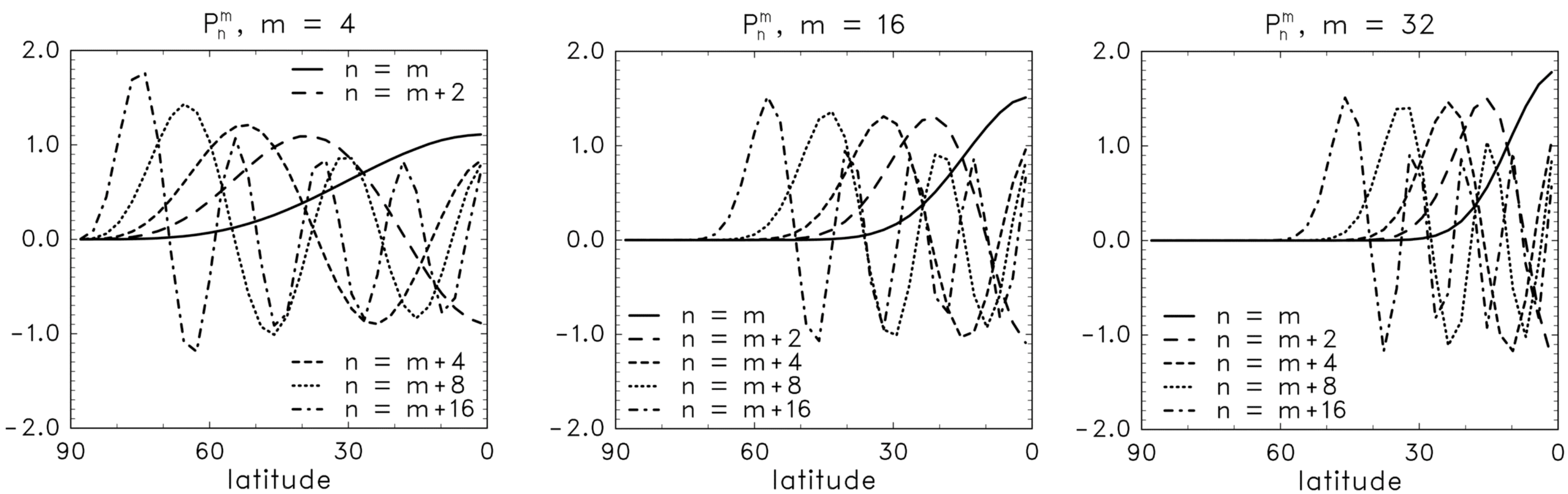
$$H_n^m = (1-\mu^2)\frac{dP_n^m}{d\mu}$$

$$\left\{\frac{\partial}{\partial\mu}\left[(\zeta+f)U\right]\right\}_n^m = -\sum_{j=1}^{J}\left[(\zeta+f)U\right]_j^m \frac{H_n^m(\mu_j)}{(1-\mu_j^2)}w_j$$

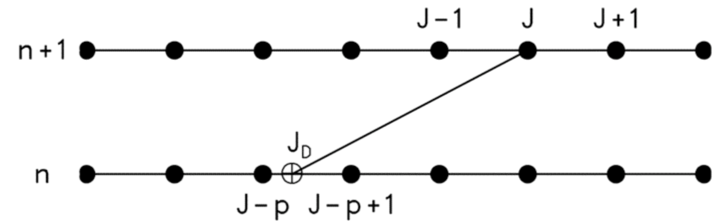
$$
\left\{\nabla^2\left[gh+\frac{(U)^2+(V)^2}{2\cos^2\varphi}\right]\right\}_n^m =
\frac{-n(n+1)}{a^2}\sum_{j=1}^{J}\left[gh+\frac{(U)^2+(V)^2}{2\cos^2\varphi}\right]_j^m P_n^m(\mu_j)w_j
$$

REDUCED GRID





SEMI-LAGRANGIAN METHOD



$$\frac{\partial q}{\partial t} + U\frac{\partial q}{\partial x} = S$$

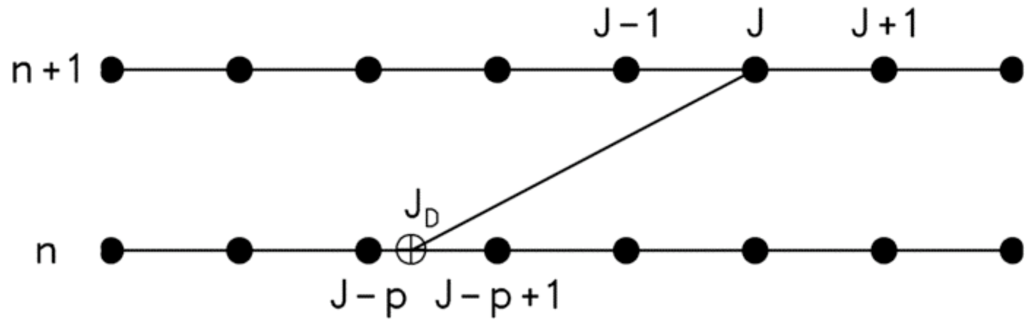
$$\frac{dq}{dt} = S(x,t) \quad , \quad \frac{dx}{dt} = U(x,t)$$

$$q_j^{n+1} - q_{j_D}^n = \int_{(x_{j_D},t^n)}^{(x_j,t^{n+1})} S(x,t)ds \quad , \quad x_j - x_{j_D} = \int_{(x_{j_D},t^n)}^{(x_j,t^{n+1})} U(x,t)ds$$

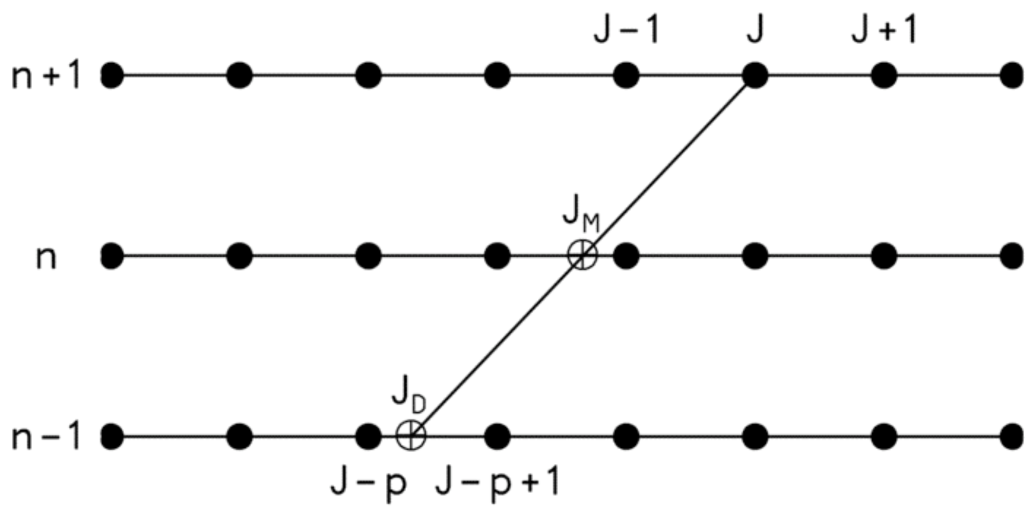
$$q_j^{n+1} = q_{j_D}^n + \Delta t S_{j_M}^{n+1/2} \quad , \quad x_{j_D} = x_j - U\Delta t$$

$$q_A^{n+1} = q_D^n + \Delta t S_M^{n+1/2} \quad , \quad x_D = x_A - U\Delta t$$

TWO-TIME LEVEL



THREE-TIME LEVEL



SHALLOW WATER EQUATIONS

$$\frac{\partial \zeta}{\partial t} = -\frac{1}{a\cos^2\varphi}\frac{\partial}{\partial \lambda}\left[(\zeta + f)\,U\right] - \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left[(\zeta + f)\,V\right]$$

$$\frac{\partial \delta}{\partial t} = \frac{1}{a\cos^2\varphi}\frac{\partial}{\partial \lambda}\left[(\zeta + f)\,V\right] - \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left[(\zeta + f)\,U\right] - \nabla^2\left[gh + \frac{U^2 + V^2}{2\cos^2\varphi}\right]$$

$$\frac{\partial u}{\partial t} = -\mathbf{v}\cdot\nabla u + \left(f + \frac{u}{a}\tan\varphi\right)v - \frac{g}{a\cos\varphi}\frac{\partial h}{\partial \lambda}$$

$$\frac{\partial v}{\partial t} = -\mathbf{v}\cdot\nabla v - \left(f + \frac{u}{a}\tan\varphi\right)u - \frac{g}{a}\frac{\partial h}{\partial \varphi}$$

$$\frac{\partial u}{\partial t} = -\mathbf{v} \cdot \nabla u + \left(f + \frac{u}{a} \tan \varphi \right) v - \frac{g}{a \cos \varphi} \frac{\partial h}{\partial \lambda}$$

$$\frac{\partial v}{\partial t} = -\mathbf{v} \cdot \nabla v - \left(f + \frac{u}{a} \tan \varphi \right) u - \frac{g}{a} \frac{\partial h}{\partial \varphi}$$

$$u^{n+1} = u^{n-1} - 2\Delta t \left\{ A_u^n \right\}$$

$$v^{n+1} = v^{n-1} - 2\Delta t \left\{ A_v^n \right\}$$

$$u^{n+1} = u^{n-1} - 2\Delta t \left\{ A_u^n \right\} \quad , \quad v^{n+1} = v^{n-1} - 2\Delta t \left\{ A_v^n \right\}$$

$$\zeta^{n+1} = \frac{1}{a \cos\varphi} \left[\frac{\partial v^{n+1}}{\partial \lambda} - \frac{\partial}{\partial \varphi} \left(u^{n+1} \cos\varphi \right) \right]$$

$$\delta^{n+1} = \frac{1}{a \cos\varphi} \left[\frac{\partial u^{n+1}}{\partial \lambda} + \frac{\partial}{\partial \varphi} \left(v^{n+1} \cos\varphi \right) \right]$$

$$\zeta^{n+1} = \zeta^{n-1} - 2\Delta t \left\{ A_\zeta^n \right\}$$

$$\delta^{n+1} = \delta^{n-1} - 2\Delta t \left\{ A_\delta^n \right\}$$

$$\frac{du}{dt} = \left(f + \frac{u}{a}\tan\varphi\right)v - \frac{g}{a\cos\varphi}\frac{\partial h}{\partial\lambda}$$

$$\frac{dv}{dt} = -\left(f + \frac{u}{a}\tan\varphi\right)u - \frac{g}{a}\frac{\partial h}{\partial\varphi}$$

$$u_{_A}^{n+1} = u_{_D}^{n-1} - 2\Delta t\left\{B_u^n\right\}_{_M}$$

$$v_{_A}^{n+1} = v_{_D}^{n-1} - 2\Delta t\left\{B_v^n\right\}_{_M}$$

SPHERICAL COORDINATES

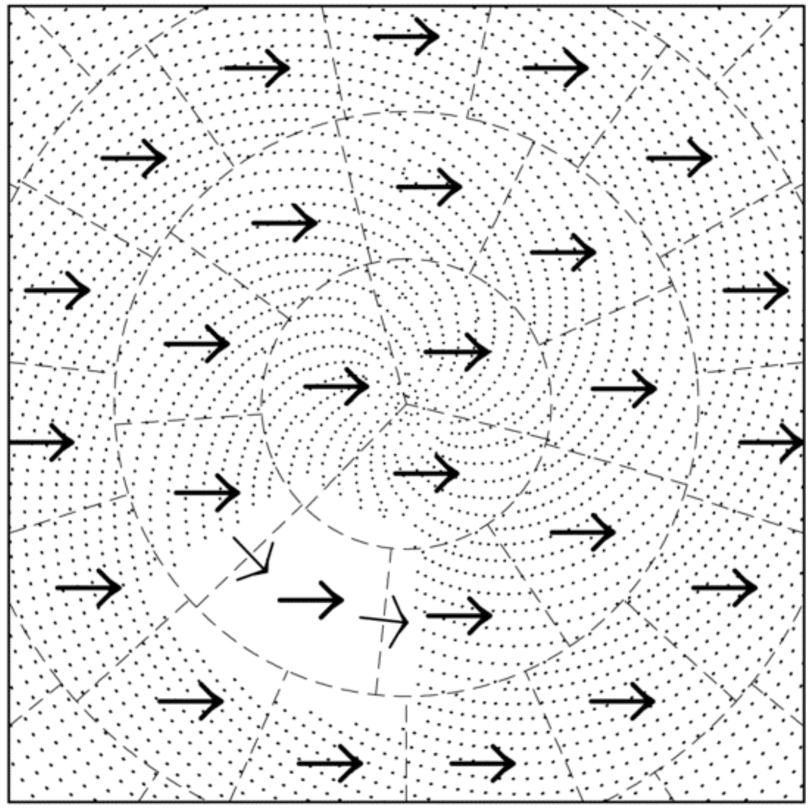
$$\frac{dx}{dt} = U(x,t) \quad , \quad x_D = x_A - 2\Delta t U(x_M)$$

$$a \cos\varphi \frac{d\lambda}{dt} = u$$

$$a \frac{d\varphi}{dt} = v$$

$$\lambda_D = \lambda_A - 2\Delta t \; \frac{u(\lambda_M, \varphi_M)}{\cos\varphi_M}$$

$$\varphi_D = \varphi_A - 2\Delta t \; v(\lambda_M, \varphi_M)$$



VECTOR FORM

$$\frac{du}{dt} = \left(f + \frac{u}{a}\tan\varphi\right)v - \frac{g}{a\cos\varphi}\frac{\partial h}{\partial\lambda}$$

$$\frac{dv}{dt} = -\left(f + \frac{u}{a}\tan\varphi\right)u - \frac{g}{a}\frac{\partial h}{\partial\varphi}$$

$$\frac{d\mathbf{v}}{dt} = -f\hat{\mathbf{k}} \times \mathbf{v} - g\nabla h$$

$$\mathbf{v}_{_A}^{n+1} = \mathbf{v}_{_D}^{n-1} - 2\Delta t\left(f\hat{\mathbf{k}} \times \mathbf{v} + g\nabla h\right)_{_M}^{n}$$

$$\mathbf{v}_{A}^{n+1} = \mathbf{v}_{D}^{n-1} - 2\Delta t \left(f\hat{\mathbf{k}} \times \mathbf{v} + g\nabla h \right)_{M}^{n}$$

$$\begin{aligned} u_{A}^{n+1}\hat{\mathbf{i}}_{A} + v_{A}^{n+1}\hat{\mathbf{j}}_{A} = {} & u_{D}^{n-1}\hat{\mathbf{i}}_{D} + v_{D}^{n-1}\hat{\mathbf{j}}_{D} \\ & - 2\Delta t \left[- \left(f + \frac{u}{a}\tan\varphi \right) + \frac{g}{a\cos\varphi}\frac{\partial h}{\partial \lambda} \right]_{M}^{n} \hat{\mathbf{i}}_{M} \\ & - 2\Delta t \left[\left(f + \frac{u}{a}\tan\varphi \right) u + \frac{g}{a}\frac{\partial h}{\partial \varphi} \right]_{M}^{n} \hat{\mathbf{j}}_{M} \end{aligned}$$

$$\hat{\mathbf{i}}_{D} = \alpha_{D}^{u}\hat{\mathbf{i}}_{A} + \beta_{D}^{u}\hat{\mathbf{j}}_{A} \quad , \quad \hat{\mathbf{j}}_{D} = \alpha_{D}^{v}\hat{\mathbf{i}}_{A} + \beta_{D}^{v}\hat{\mathbf{j}}_{A}$$

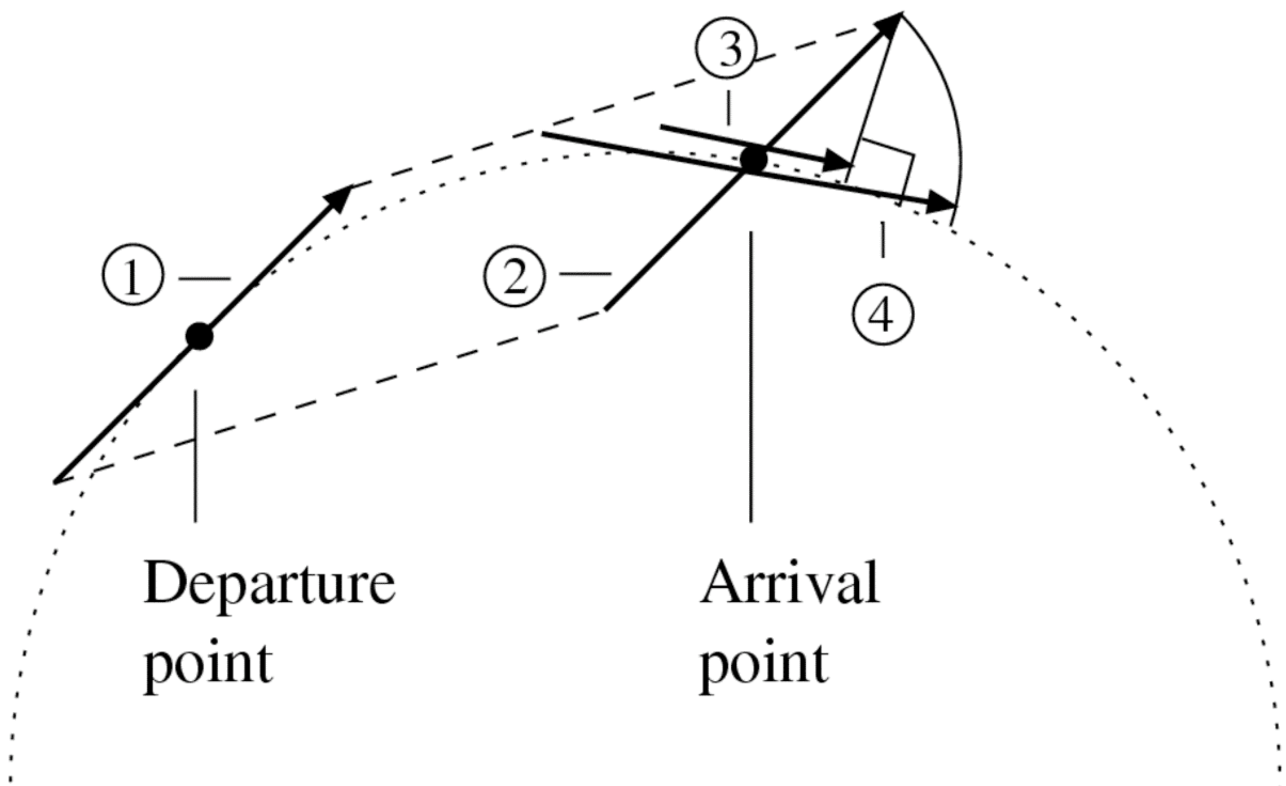
$$\hat{\mathbf{i}}_{M} = \alpha_{M}^{u}\hat{\mathbf{i}}_{A} + \beta_{M}^{u}\hat{\mathbf{j}}_{A} \quad , \quad \hat{\mathbf{j}}_{M} = \alpha_{M}^{v}\hat{\mathbf{i}}_{A} + \beta_{M}^{v}\hat{\mathbf{j}}_{A}$$

$$
\begin{aligned}
u_A^{n+1}\hat{\mathbf{i}}_A + v_A^{n+1}\hat{\mathbf{j}}_A &= u_D^{n-1}\hat{\mathbf{i}}_D + v_D^{n-1}\hat{\mathbf{j}}_D \\
&- 2\Delta t\left[-\left(f + \frac{u}{a}\tan\varphi\right) + \frac{g}{a\cos\varphi}\frac{\partial h}{\partial\lambda}\right]_M^n \hat{\mathbf{i}}_M \\
&- 2\Delta t\left[\left(f + \frac{u}{a}\tan\varphi\right)u + \frac{g}{a}\frac{\partial h}{\partial\varphi}\right]_M^n \hat{\mathbf{j}}_M
\end{aligned}
$$

$$
\begin{aligned}
u_A^{n+1}\hat{\mathbf{i}}_A + v_A^{n+1}\hat{\mathbf{j}}_A &= u_D^{n-1}\left(\alpha_D^u\hat{\mathbf{i}}_A + \beta_D^u\hat{\mathbf{j}}_A\right) \\
&+ v_D^{n-1}\left(\alpha_D^v\hat{\mathbf{i}}_A + \beta_D^v\hat{\mathbf{j}}_A\right) \\
&- 2\Delta t\left[-\left(f + \frac{u}{a}\tan\varphi\right) + \frac{g}{a\cos\varphi}\frac{\partial h}{\partial\lambda}\right]_M^n \left(\alpha_M^u\hat{\mathbf{i}}_A + \beta_M^u\hat{\mathbf{j}}_A\right) \\
&- 2\Delta t\left[\left(f + \frac{u}{a}\tan\varphi\right)u + \frac{g}{a}\frac{\partial h}{\partial\varphi}\right]_M^n \left(\alpha_M^v\hat{\mathbf{i}}_A + \beta_M^v\hat{\mathbf{j}}_A\right)
\end{aligned}
$$

$$
u_A^{n+1} = u_D^{n-1}\alpha_D^u + v_D^{n-1}\alpha_D^v - 2\Delta t\left\{\text{Stuff}\right\}_M
$$

$$
v_A^{n+1} = v_D^{n-1}\beta_D^v + u_D^{n-1}\beta_D^u - 2\Delta t\left\{\text{Other Stuff}\right\}_M
$$



Williamson, Olson and Jablonowski, 2008, Two dynamical
core formulation flaws …, Mon. Wea. Rev, *sub judice*

EFFICIENCY GAINS SINCE 1987

	NWP	CLIMATE
Eulerian $\rightarrow$ semi-Lagrangian	5	1.2
3-time-level $\rightarrow$ 2-time-level	1.8	1.3
Full grid $\rightarrow$ Reduced grid	1.4	1.4
Quadratic $\rightarrow$ Linear grid	3.4	3.4
Troposphere $\rightarrow$ Stratosphere Thinner PBL levels	1.7	1.7
	73	13

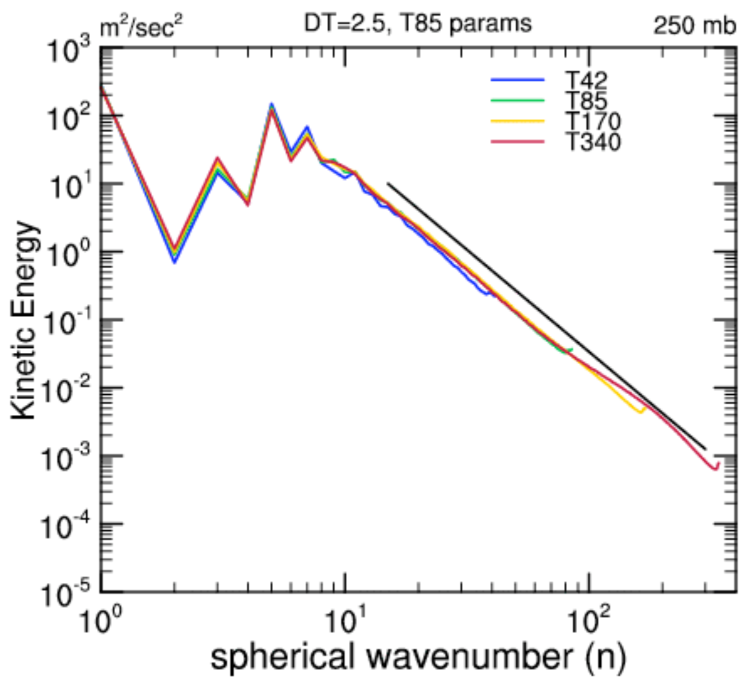
For climate: T75 for the cost of T42

Machenhauer, B., 1979: The spectral method. In A. Kasahara (ed.), *Numerical Methods Used in Atmospheric Models, Vol. 2*, GARP Publications Series No 17, WMO and ICSU, Geneva, pp. 121–275.

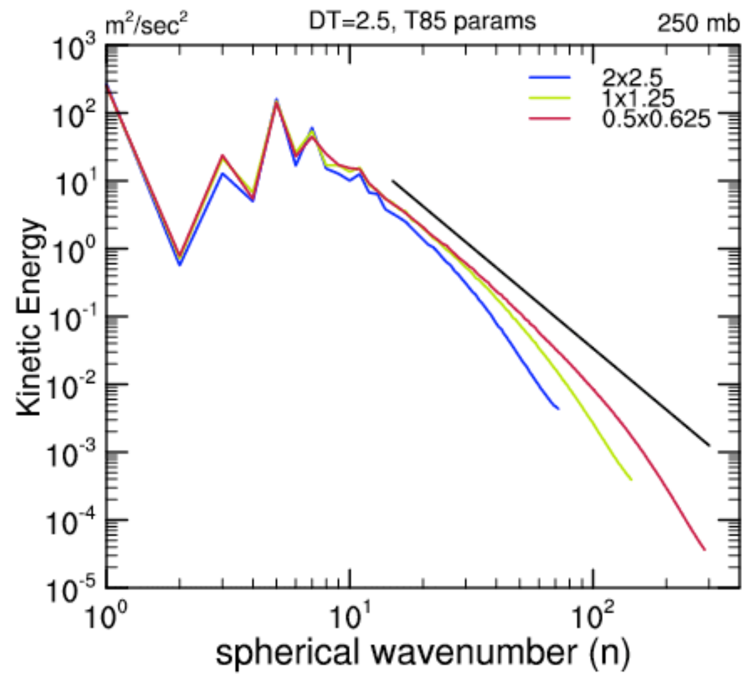
Williamson, D. L. and R. Laprise, 2000: Numerical Approximations for Atmospheric General Circulation Models. In P. Mote and A. O'Neill (eds.), *Numerical Modelling of the Global Atmosphere in the Climate System*, Kluwer Academic Publishers, Netherlands, 127–219.

Staniforth, A. and Côté, J., 1991: Semi-Lagrangian integration schemes for atmospheric models - A Review, *Mon. Wea. Rev.*, **119**, 2206–2223.

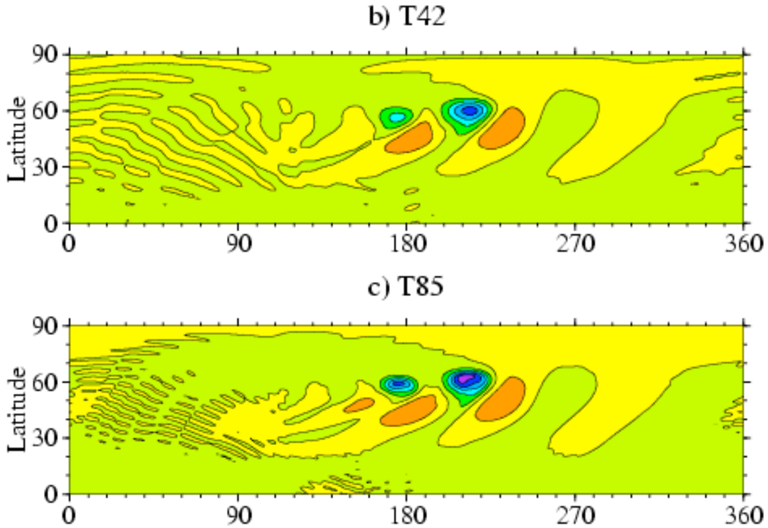
EULERIAN SPECTRAL



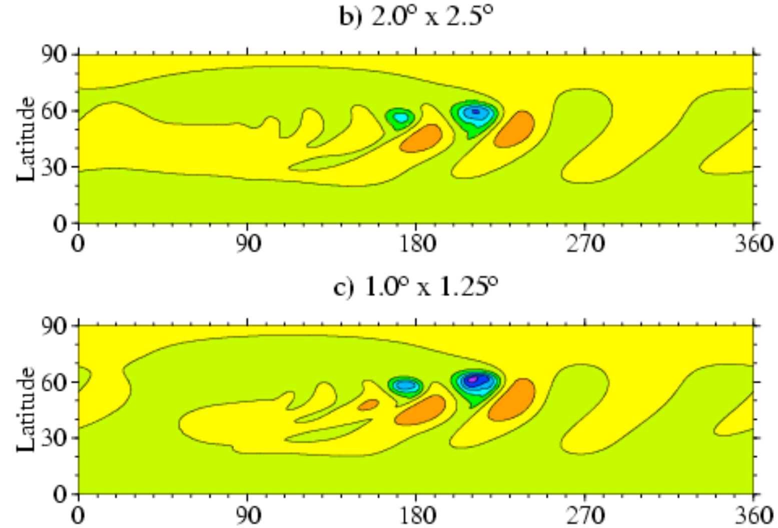
FINITE VOLUME



EULERIAN SPECTRAL

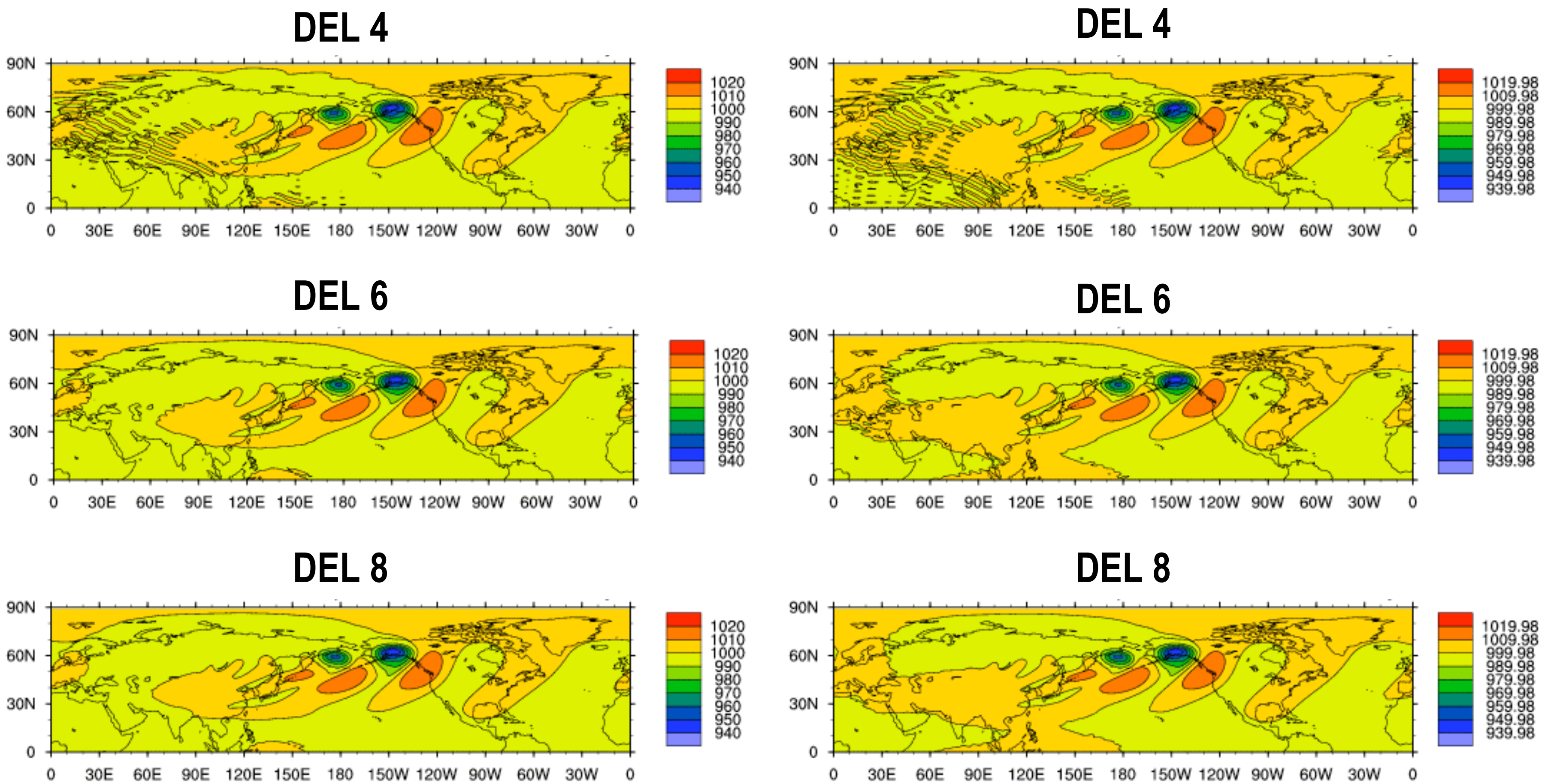


FINITE VOLUME

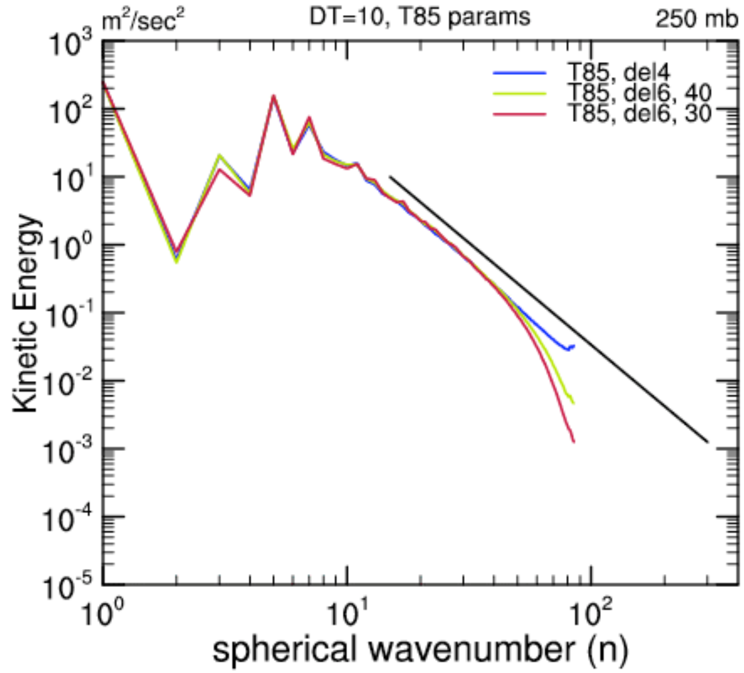


From Figures 5.5 and 5.7 of Jablonowski and Williamson, NCAR TN-469+STR, 2006

T85 EULERIAN SPECTRAL
DAY 9



EULERIAN SPECTRAL



FINITE VOLUME