Vertical discretizations

Thursday 5 June, 2008

Outline

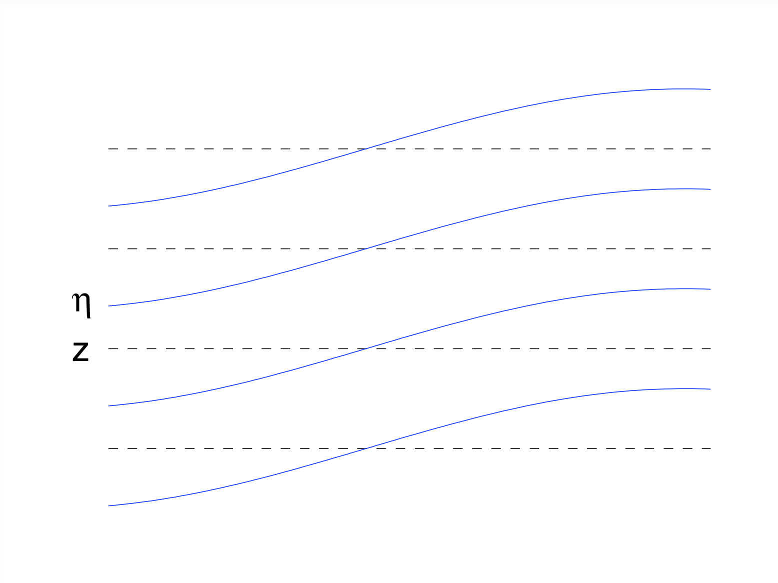
- Different vertical coordinates

- Bottom and top boundary conditions

- Example: the energy and angular momentum conserving scheme of Simmons and Burridge

- Wave dispersion and balance

Different vertical coordinates



Transformation rules:

$$\frac{\partial\psi}{\partial z} = \frac{\partial\eta}{\partial z}\frac{\partial\psi}{\partial\eta}$$

$$\left(\frac{\partial\psi}{\partial s}\right)_z = \left(\frac{\partial\psi}{\partial s}\right)_\eta + \left(\frac{\partial\psi}{\partial\eta}\right)\left(\frac{\partial\eta}{\partial s}\right)_z$$

$$\frac{D\psi}{Dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}.\nabla_{\mathrm{H}} + \dot{\eta}\frac{\partial\psi}{\partial\eta}$$

Note it is usual to retain the usual velocity components

$$u = \dot{\lambda}a\cos\phi, \quad v = \dot{\phi}a, \quad w = \dot{z} \text{ (as well as } \dot{\eta} \text{)}$$

Examples

(i) **Height** $\eta = z$

(ii) **Pressure** $\eta = p$

(iii) **Mass** $\eta = \int_z^\infty g\rho dz$

Examples

(iv) **Terrain following** variants

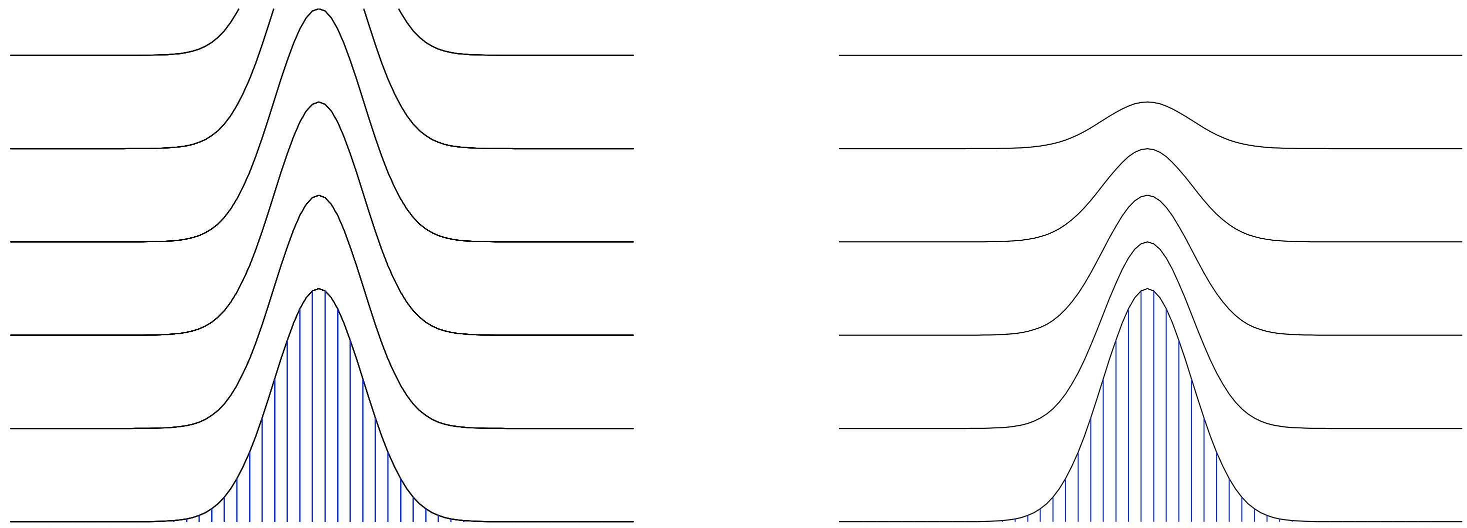
E.g. $\eta = z - z_s$,

or $\eta = p/p_s$

Also (v) **hybrid terrain following** variants

E.g. $\eta = a + b$

where $p = ap_0 + bp_s$



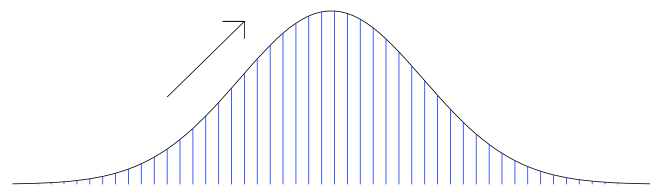
Examples

(vi) **Isentropic coordinate** $\eta = f(\theta)$

(vii) **Lagrangian vertical coordinate** $\dot{\eta} = 0$

Bottom boundary condition

No normal flow



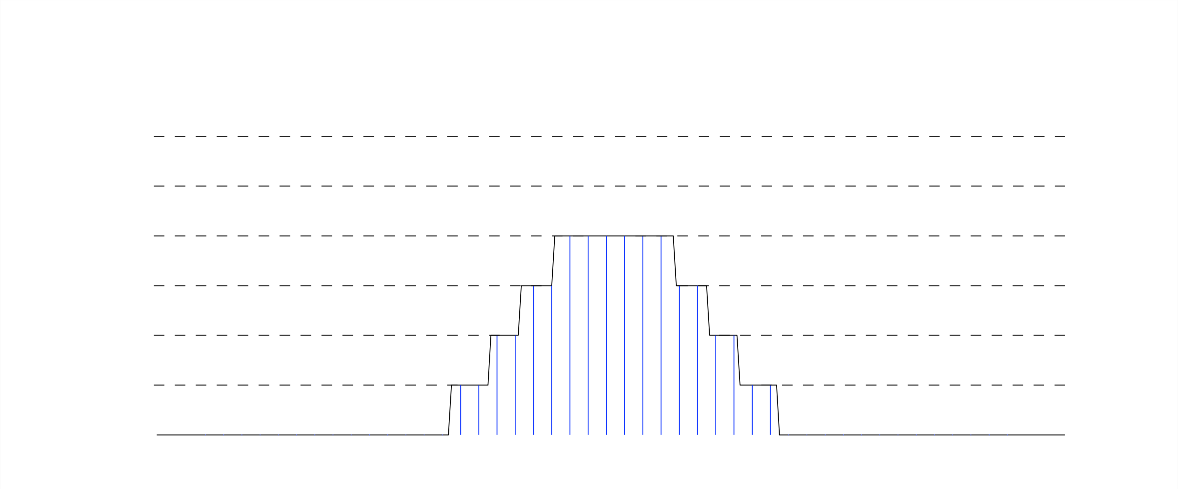
$\dot{\eta} = 0$ if η is terrain following

$w = 0$ if the ground is flat. Otherwise $w = \mathbf{v}.\nabla_H z_s$

When a boundary layer is included (**no slip**) then $\mathbf{v} = \mathbf{0}$, $w = 0$

But for a frictionless dynamical core (**free slip**) we need a value of $\mathbf{v}_s$.

An alternative is to use a **terrain intersecting coordinate**



possibly using **fractional cells** or **shaved cells**

Tricky when the coordinate surfaces and their intersections with the ground may move.

Top boundary condition

The real atmosphere has no top boundary!

Practical choices include

• **rigid lid** at some constant z. $w = 0$ there. Angular momentum and energy conservation are preserved.

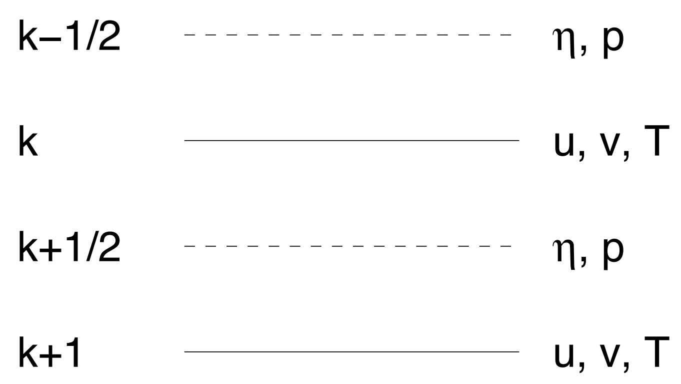
• **elastic lid** at some constant p (may be $p = 0$). $\dot{p} = 0$ there. Continuous equations conserve angular momentum and enthalpy.

It is common to include some (non-scale-selective) damping near the model top to reduce **spurious wave reflection**.

Example: the Simmons and Burridge energy and angular momentum conserving scheme

for the hydrostatic primitive equations

Hybrid "sigma-pressure" coordinate $p = ap_0 + bp_s$



Hydrostatic equation

$$\frac{\partial \Phi}{\partial \eta} = -\frac{RT}{p}\frac{\partial p}{\partial \eta}$$

naturally becomes

$$\Phi_{k+1/2} - \Phi_{k-1/2} = -RT_k \ln \frac{p_{k+1/2}}{p_{k-1/2}}$$

and then

$$\Phi_k = \Phi_{k+1/2} + \alpha_k RT_k$$

Note the existence of a **'computational mode'**: a non-zero pattern in T that is invisible to the Φ_k.

Angular momentum conservation

$$\frac{D\mathbf{v}}{Dt} + f\hat{\mathbf{z}} \times \mathbf{v} + \nabla\Phi + \frac{RT}{p}\nabla p = 0$$

Take $\int_0^1 (\ldots)\, \partial p/\partial\eta \, d\eta$

For angular momentum conservation we require

$$\sum_{r=1}^{N} \Phi_r \frac{\partial}{\partial\lambda}\Delta p_r = \Phi_s \frac{\partial p_s}{\partial\lambda} + \sum_{r=1}^{N} R\left(\frac{T}{p}\frac{\partial p}{\partial\lambda}\right)_r \Delta p_r$$

which is satisfied if

$$\left(\frac{RT}{p}\nabla p\right)_k = \frac{rT_k}{\Delta p_k}\left[\left(\ln\frac{p_{k+1/2}}{p_{k-1/2}}\right)\nabla p_{k-1/2} + \alpha_k \nabla(\Delta p_k)\right]$$

Energy conservation

$\mathbf{v}$ times momentum equation:

$$\frac{D}{Dt}\left(\frac{\mathbf{v}^2}{2}\right) + \mathbf{v}.\nabla\Phi + \frac{RT}{p}\mathbf{v}.\nabla p = 0$$

Thermodynamic equation

$$\frac{D}{Dt}c_pT - \frac{RT\omega}{p} = 0$$

where

$$\omega \equiv \dot{p} = -\int_0^\eta \nabla.\left(\mathbf{v}\frac{\partial p}{\partial \eta}\right)\, d\eta + \mathbf{v}.\nabla p$$

We require

$$\int \left(\mathbf{v}.\nabla\Phi + \frac{RT}{p}\mathbf{v}.\nabla p - \frac{RT\omega}{p} \right) \frac{\partial p}{\partial \eta} \, d\eta \, dA = 0$$

This will be satisfied provided we evaluate

$$\left(\frac{1}{p}\nabla p \right)_k$$

as in the momentum equation, and RT/p times the vertical integral term as

$$\frac{RT_k}{\Delta p_k} \left[\left(\ln \frac{p_{k+1/2}}{p_{k-1/2}} \right) \sum_{r=1}^{k-1} \nabla.(\mathbf{v}_r \Delta p_r) + \alpha_k \nabla.(\mathbf{v}_k \Delta p_k) \right]$$

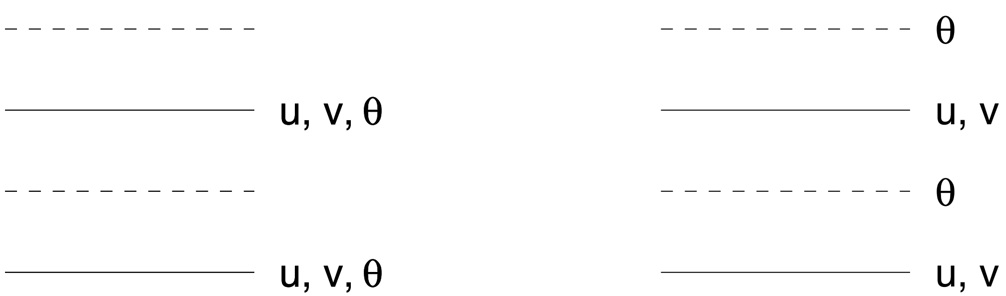
Wave propagation, and balance

There are issues analogous to those that arise when choosing horizontal discretizations:

Which choices of prognostic variables and grid staggering best capture **wave propagation**, **adjustment**, and **balance**?

The Lorenz and Charney-Phillips grids

for the hydrostatic case



The Lorenz grid computational mode

Consider the Simmons and Burridge scheme

Linearize about a state of hydrostatic balance, and compare neighboring Φ values

$$\Phi'_k - \Phi'_{k-1} = (\alpha_k - (\Delta \ln p)_k)T'_k - \alpha_{k-1}T'_{k-1}$$

The left hand side can vanish provided a certain average of the T's vanishes

There is some oscillatory pattern in T_k that is invisible to the Φ_k, and therefore does not propagate.

This also implies a spurious resonant response to steady forcing.

Compressible Euler equations

We now need **five** prognostic variables, so there are many choices of staggering.

Also, there is some freedom over which thermodynamic variables to use (any two from ρ, p, T, θ, etc...)

Here consider z **coordinate**. Similar reasoning applies to other vertical coordinates (e.g. mass, isentropic).

We want to minimize the use of vertical averaging and of finite differences over $2\Delta z$

Numerical exploration of a large number of cases shows

• accurate representation of acoustic waves is necessary (but not sufficient) for an accurate representation of inertio-gravity waves

• which in turn is necessary (but not sufficient) for an accurate representation of Rossby waves.

Acoustic waves

$$\omega^2 \approx m^2 c^2$$

To capture acoustic waves (reasonably) accurately in the limit of short vertical wavelength we require

- $\delta_z p$ at the same level as w
- $\delta_z w$ at the same level as p

What if we don't predict p ?

Inertio-gravity waves

$$\omega^2 \approx \frac{m^2 f^2 + K^2 N^2}{m^2 + K^2}$$

To capture inertio-gravity waves (reasonably) accurately in the limit of short vertical wavelength we require

- u and v at the same level as p
- buoyancy variable at the same level as w

What if we don't predict p ?

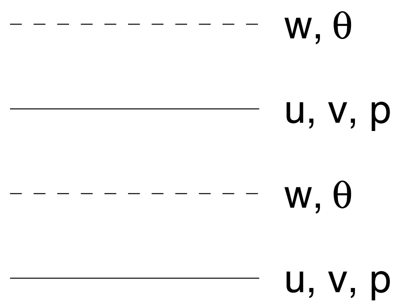
Rossby waves

$$\omega \approx -\frac{k\beta N^2}{m^2 f^2 + K^2 N^2}$$

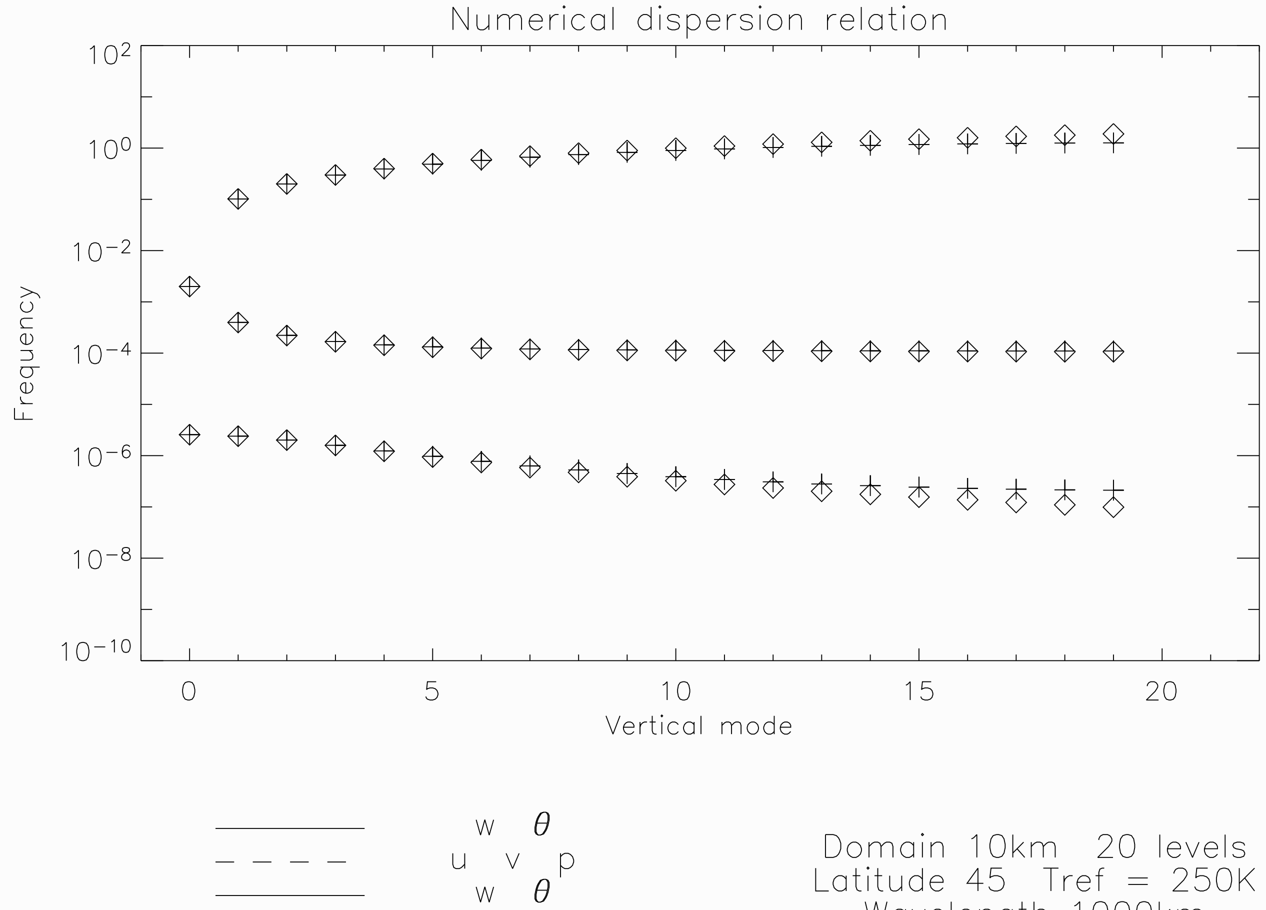
To capture Rossby waves (reasonably) accurately in the limit of short vertical wavelength we require

- buoyancy variable at the same level as w

Example: optimal configuration



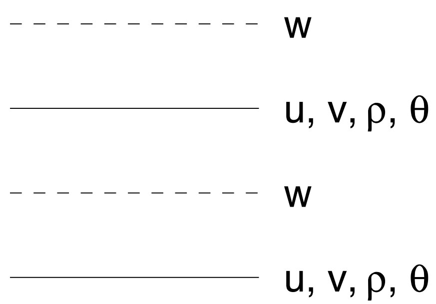
with pressure gradient term written as $c_p \theta \nabla \Pi$



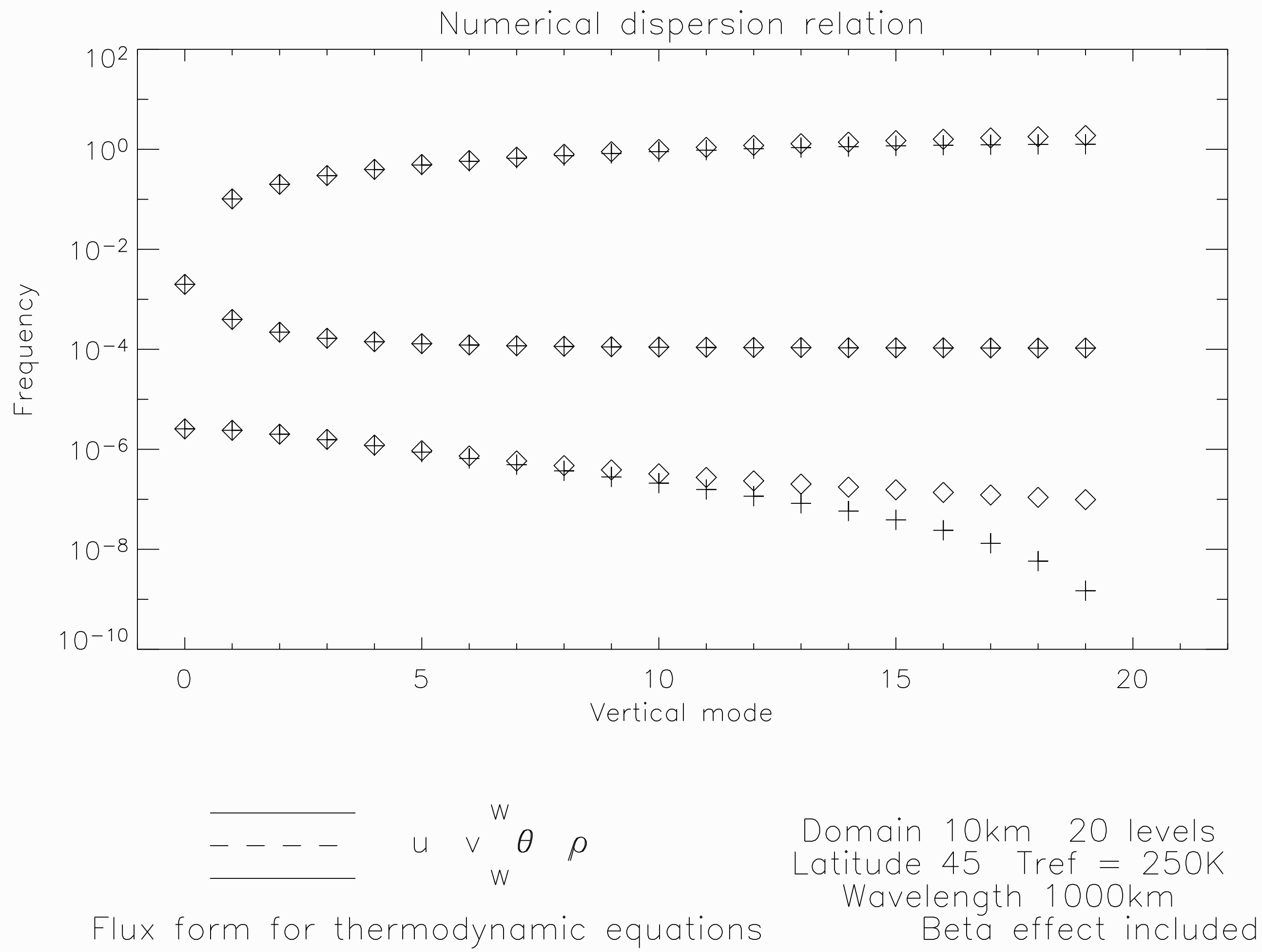
Flux form for thermodynamic equations

Domain 10km 20 levels
Latitude 45 Tref = 250K
Wavelength 1000km
Beta effect included

Example: Lorenz-like grid



Note computational mode.


Numerical dispersion relation
Frequency
Vertical mode
10^2
10^0
10^-2
10^-4
10^-6
10^-8
10^-10
0
5
10
15
20
w
u v θ ρ
w
Flux form for thermodynamic equations
Domain 10km 20 levels
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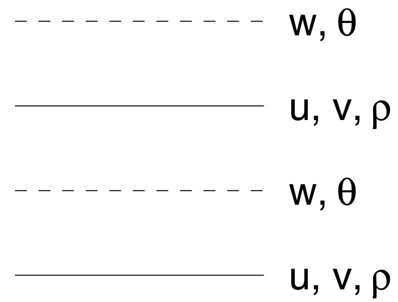
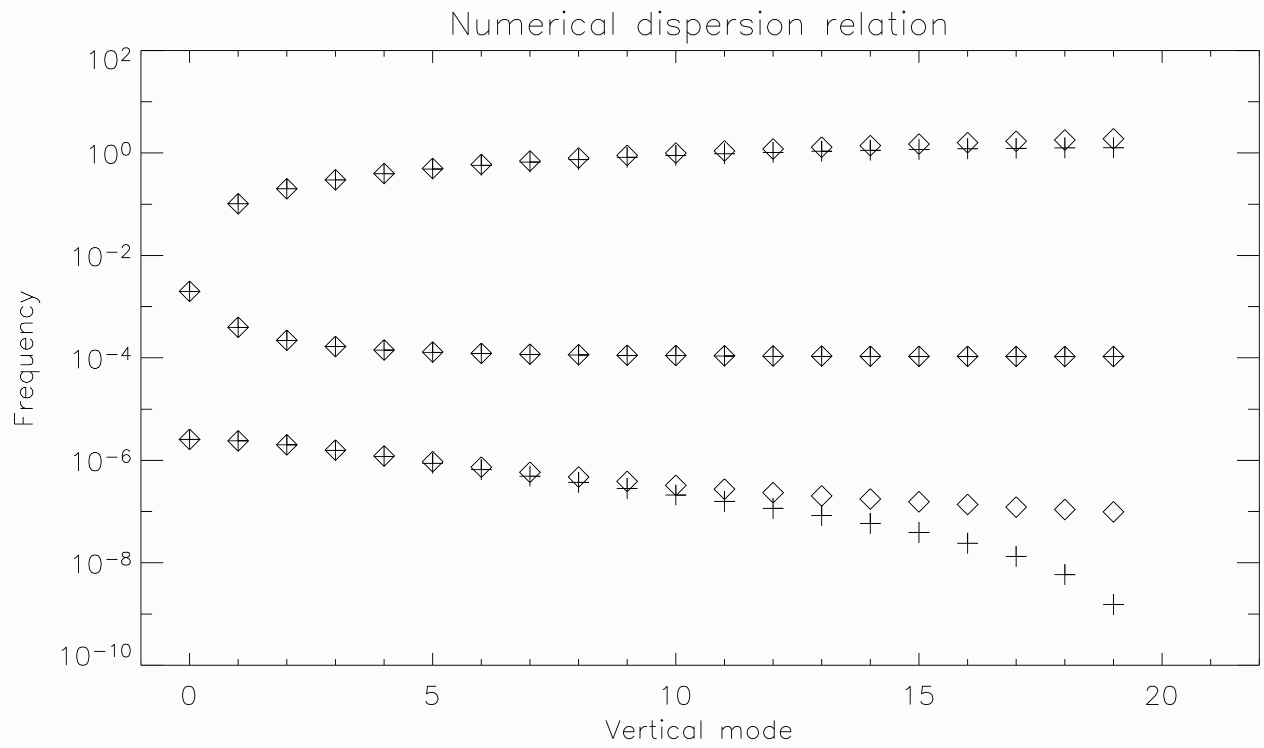
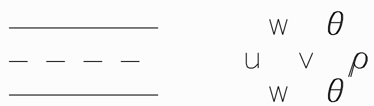
Example: sub-optimal configuration



with pressure gradient term written as $(1/\rho)\nabla p$ and p diagnosed from ρ and θ





Flux form for thermodynamic equations

Domain 10km 20 levels
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Wavelength 1000km
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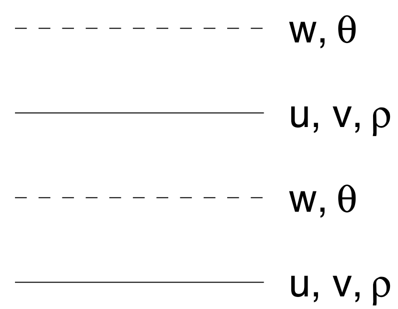
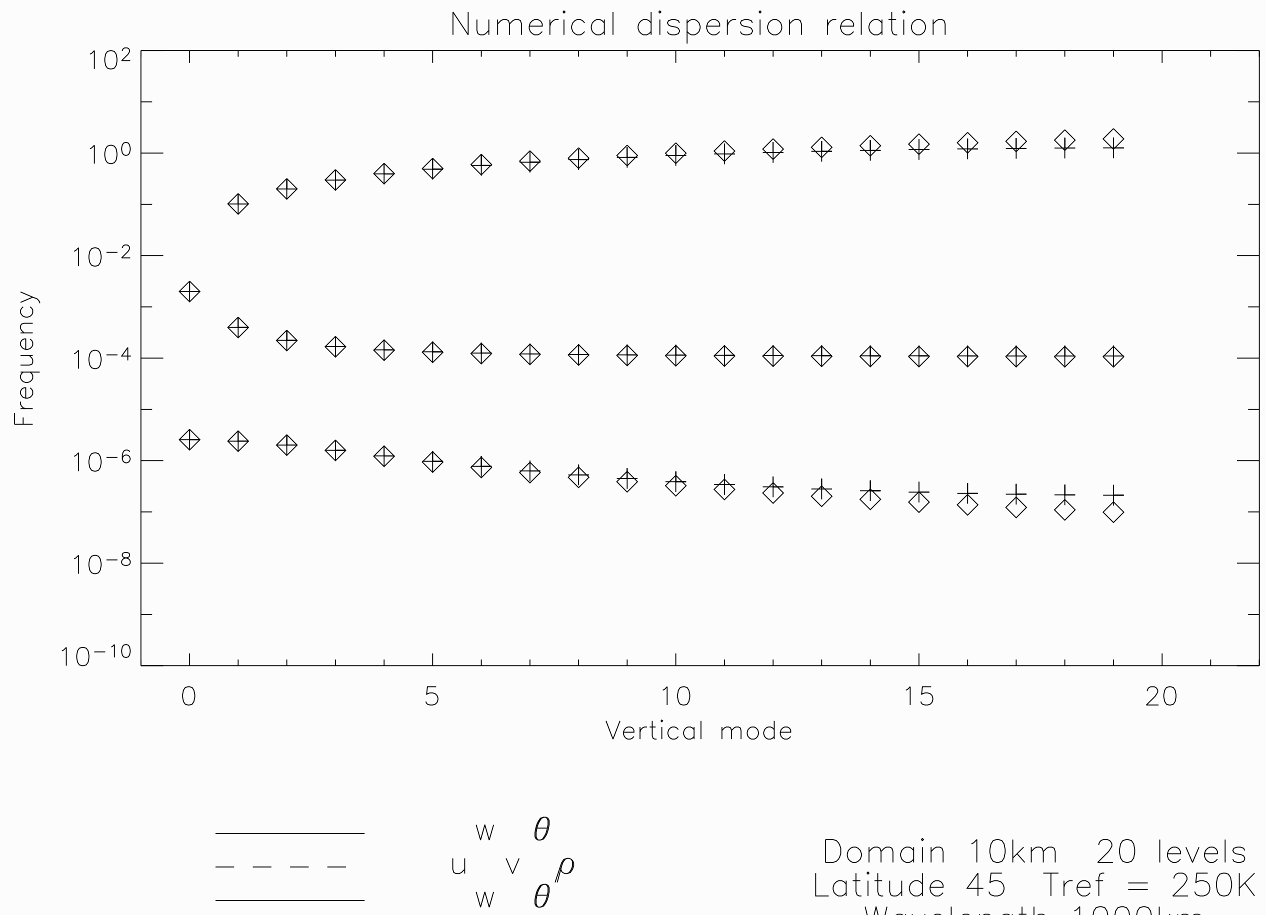
Example: optimal configuration predicting ρ



with pressure gradient term written as $c_p\theta\nabla\Pi$ and Π diagnosed from ρ and θ



Flux form for thermodynamic equations

Domain 10km 20 levels
Latitude 45 Tref = 250K
Wavelength 1000km
Beta effect included

Note the **apparent incompatibility** between **optimal wave propagation without computational modes**

and

energy conservation

But see Friday's lecture on "conservation"...

The ρ-θ-q conundrum

- Optimal wave propagation requires ρ staggered with respect to θ
- Conservation of moisture requires ρ collocated with q
- Physical parameterizations require q collocated with θ