Conservation

in numerical model dynamical cores

Friday 6 June, 2008

Outline

- Conservation properties of the continuous adiabatic frictionless governing equations
- What conservation properties can we obtain in numerical models?
- Which conservation properties are most relevant/important?

  Finite resolution effects; the adiabatic frictionless **limit**

  Respecting appropriate asymptotic limits

  Spurious sources vs physical sources

(Closely following T 2008, J. Comput. Phys.)

Conservation properties of the continuous adiabatic frictionless governing equations

Flux form conservation laws

Lagrangian conservation laws

Conserved integral quantities

Kinematic identities

Flux form conservation laws

$$\frac{\partial A}{\partial t} + \nabla.\mathbf{F} = 0,$$

Quantity	A	$\mathbf{F}$
Mass	ρ	$\rho\mathbf{u}$
Angular momentum	$\rho\widehat{\mathbf{z}}.\left[\mathbf{r}\times(\mathbf{u}+\mathbf{\Omega}\times\mathbf{r})\right]$	$\mathbf{u}A + p\widehat{\mathbf{z}}\times\mathbf{r}$
Energy	$\rho\left(\frac{1}{2}\mathbf{u}^2 + c_v T + \Phi\right)$	$\mathbf{u}\,(A+p)$

Lagrangian conservation laws

$$\frac{D\chi}{Dt} = 0 \quad \Rightarrow \quad \frac{Df(\chi)}{Dt} = 0$$

Potential temperature θ

Potential vorticity $Q = \boldsymbol{\zeta}.\nabla\theta/\rho$

Specific tracer q or **tracer mixing ratio** η

Each Lagrangian conservation law generates an infinite family of flux form conservation laws

$$\frac{\partial \rho f(\chi)}{\partial t} + \nabla.\left(\rho \mathbf{u} f(\chi)\right) = 0$$

Conserved integral quantities

Mass per unit θ in an isentropic layer	$\mathcal{F}(\theta) = \int \rho / \lvert\nabla\theta\rvert \, dA$
Mass per unit θ in an isentropic layer within a material contour	$\mathcal{M} = \int_D \rho / \lvert\nabla\theta\rvert \, dA$
Absolute circulation around an isentropic material contour	$\mathcal{C} = \oint_\Gamma \mathbf{v}_a . d\mathbf{r} = \int_D \rho Q / \lvert\nabla\theta\rvert \, dA$

Kinematic identities

The global integrals of horizontal divergence

$$\int_D \delta \, dA$$

and vertical component of vorticity

$$\int_D \zeta \, dA$$

must vanish on any isosurface of the vertical coordinate that wraps the sphere.

Techniques for obtaining or approximating conservation properties in numerical models

1 Predict the desired variable using a discrete flux form conservation law

$$\frac{A_j^{n+1} - A_j^n}{\Delta t} + \frac{F_{j+1/2} - F_{j-1/2}}{\Delta x_j} = 0$$

E.g. A might be ρ times specific humidity.

Techniques for obtaining or approximating conservation properties in numerical models

2 Impose discrete analogues of special cancellations

E.g. Coriolis terms on the C-grid; Arakawa Jacobian

In some cases there are systematic ways of deriving such schemes using Poisson bracket and Nambu bracket ideas

(May only work globally)

Techniques for obtaining or approximating conservation properties in numerical models

3 Lagrangian conservation properties

Use a Lagrangian solution technique

Use Lagrangian or quasi-Lagrangian coordinates (or at least vertical coordinate)

Nonoscillatory advection schemes (perhaps combined with ‘reverse engineering’)

Techniques for obtaining or approximating conservation properties in numerical models

4 Special forms of scale-selective dissipation

E.g. Anticipated potential vorticity method

E.g. Energy backscatter

Which conservation properties are most relevant/important?

Finite resolution effects

Take $\rho \equiv 1$, let χ be specific tracer. Define

$$V_i = \int_{\text{cell } i} dV$$

$$m_i = \left(\int_{\text{cell } i} \chi \, dV \right) / V_i$$

$$r_i = \left(\int_{\text{cell } i} \chi^2 \, dV \right) / V_i$$

Then

$$\int \chi dV = \Sigma_i m_i V_i$$

But

$$\int \chi^2 dV = \Sigma_i r_i V_i \geq \Sigma_i m_i^2 V_i$$

Which conservation properties are most relevant/important?

Adiabatic and frictionless, or adiabatic frictionelss *limit* ?

Quantities like tracer variance and potential enstrophy that cascade downscale are dissipated even in the limit of vanishing viscosity and thermal diffusivity. They are **non-Robust** invariants

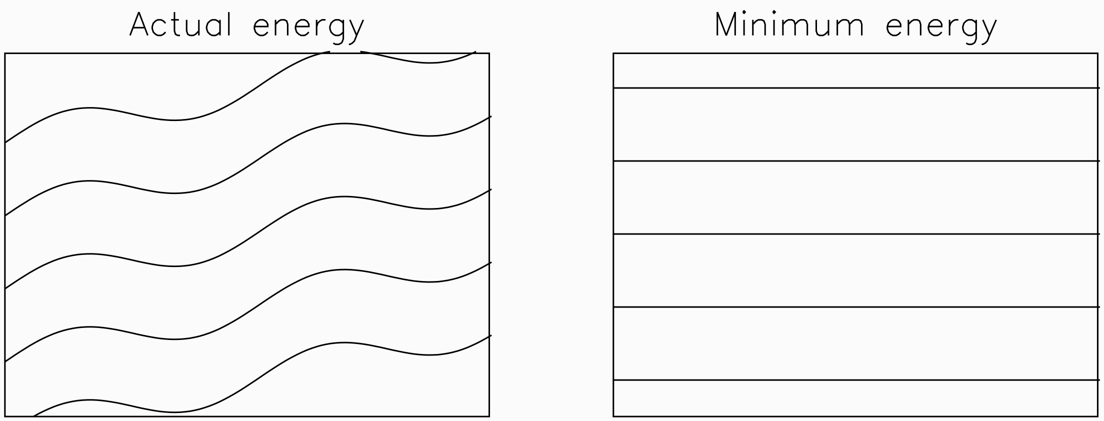
Dilemma: attempt to conserve non-robust invariants, then dissipate them with a **sub-grid model**, or use inherently dissipative numerical methods **(ILES)**.

Matlab demo of enstrophy conservation

What about **energy**?

Energy

Total energy ($\sim 3 \times 10^{9}\mathrm{Jm}^{-2}$) is made up of available and unavailable contributions



	Unavailable PE	Available PE	KE
Ratio:	2000	4	1

Unavailable and available energy are separately conserved

Unavailable energy is a function of the $\mathcal{F}(\theta)$ - almost robust

Can conserve mass in each model isentropic layer by using θ as a vertical coordinate

There is some evidence that about 5-10% of available energy cascades downscale in the free atmosphere (the rest goes upscale before being dissipated by the boundary layer)

Which conservation properties are most relevant/important?

Respecting asymptotic limits (Mike Cullen)

There is no mathematical proof that solutions of the continuous governing equations exist, but there are for certain relevant asymptotic limits (e.g. semi-geostrophic equations).

It is argued that the properties that are essential for the existence proof (e.g. boundedness of PV under advection) are also the properties that **control** the dynamics, and should therefore be respected by numerical methods.

This also argues for numerical methods respecting the mixed hyperbolic-elliptic nature of the asymptotic limits.

Which conservation properties are most relevant/important?

Spurious sources vs physical sources

Our numerical solutions should be accurate provided spurious numerical sources of conservable quantities are much weaker than true physical sources.

Conveniently expressed in terms of timescales.

Mass

Essentially no sources of dry air:

$$\tau \sim \infty$$

Momentum and angular momentum

Locally, adjustment towards balance is fast

$\tau \sim$ few 10s of seconds to few 10s of hours

But, in a zonal mean, terms in the u equation are not in geostrophic balance.

Global mean angular momentum $\sim \pm 0.4 \times 10^{26}\,\mathrm{kg\,m^2s^{-1}}$

Typical surface torque $\sim \pm 0.5 \times 10^{20}\,\mathrm{kg\,m^2s^{-2}}$

$\tau \sim 10\,\mathrm{days}$

(But locally much longer, e.g. QBO.)

Tracer variance

Estimates of “mixdown time” suggest $\tau \sim 10 - 20\,\mathrm{days}$

Potential enstrophy

Enstrophy budgets suggest $\tau \sim 10\,\mathrm{days}$

Unavailable energy

Global mean $\sim 3 \times 10^9\,\mathrm{Jm}^{-2}$

Total energy throughput of climate system $\sim 240\,\mathrm{Wm}^{-2}$

$\tau \sim 150\,\mathrm{days}$

Available energy

Global mean $\sim 6 \times 10^6 \, \mathrm{Jm}^{-2}$

Available energy throughput of the atmosphere $\sim 2 \, \mathrm{Wm}^{-2}$

$\tau \sim 30 \, \mathrm{days}$

Summary

Quantity	Robust	Cascade	Approx. timescale
Mass	Yes		Infinite
Momentum			Minutes to hours
Angular momentum			10 days (locally longer)
Potential enstrophy		Yes	10 days
Tracer variance		Yes	10 days
Unavailable energy	Almost		150 days
Available energy		Yes (5-10%)	20-30 days
Entropy	Almost		Variable

Concluding remark

There is no perfect general purpose method for solving the compressible Navier-Stokes equations.

The methods developed by weather and climate modellers aim to capture the most important aspects of the physics **for flow regime of interest**

and to exploit flow properties for efficiency.