A Little Bit More on Waves and Turbulence

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What I hope to talk about

- Complement John Thuburn’s lectures
- Discuss dissipation
- Characteristics and hyperbolicity
- Relationship to mesh refinement
- Slow ‘manifold’ and QG turbulence
- Fjortoft constraints and power laws
- Some results

Dissipation?

- Global atmosphere gets energy from sun
- Acts as a heat/steam engine to get motion i.e. weather/climate
- Motion is dissipated by viscous stresses and converted back to heat

- What is the timescale for dissipation?

NSE term $\frac{\partial}{\partial t} - \nu \nabla^2$

T depends on L $T \sim \frac{L^2}{\nu}$

For dry air $\nu = 1.5x10^{-6} m^2/s$

For L =1km T~ 2000 years
Even for L=1m T~.7 days

ON RELEVANT SCALES EQUATIONS ARE A HYPERBOLIC SYSTEM
STUDY THE CHARACTERISTICS

Method of Characteristics

Simplest Case :

First order equation

$$a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = c(x,t)$$

$$u(x,0) = F(x)$$

Use method of characteristics

$$\frac{dt}{ds} = a(x,t)\,,\;\; \frac{dx}{ds} = b(x,t),\;\text{and}\; \frac{du}{ds} = c(x,t)$$

Proof : $u(s) = u(x(s), t(s))$, use chain rule

Easy example $a = 1,\; b = const.$ and $c = 0$

$t = s, x(s) = bs$ and $u = const.$

Along each characteristic curve $x = x(o) + bt$

$$u(x,t) = F(x - bt)$$

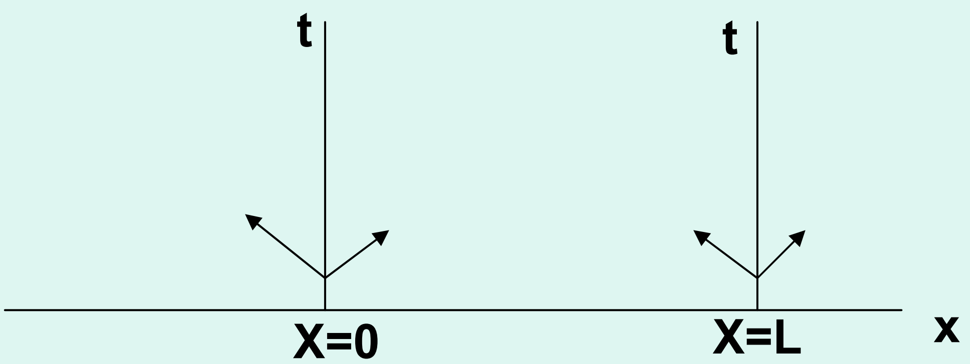
Also works for hyperbolic systems

$$\mathbf{A}(\mathbf{w},\mathbf{x},t)\frac{\partial}{\partial t}\mathbf{w}+\mathbf{B}(\mathbf{w},\mathbf{x},t)\cdot\nabla\mathbf{w}+\mathbf{C}(\mathbf{w},\mathbf{x},t)\mathbf{w}=\mathbf{F}$$

The compressible Euler equations have this form

The various wave motions (acoustic, IGW, Rossby) correspond to different characteristics. Can solve the general Initial Boundary Value Problem (IBVP)

This involves diagonalizing the system and propagating characteristic variables through the domain boundaries.



What about filter/balance approximations?

We will consider the local problem of linearized primitive equations FIRST : Why is the hydrostatic approximation made?

$$\frac{\partial}{\partial t}\mathbf{u} + U_0\frac{\partial}{\partial x}\mathbf{u} + f\mathbf{k}\mathbf{x}\mathbf{u} + \nabla_h\phi = 0\,,$$

with $\mathbf{u} = (u, v)$.

$$\frac{\partial}{\partial t}b + U_0\frac{\partial}{\partial x}b + N^2 w = 0$$

$$\nabla\cdot\mathbf{u} + w_z = 0$$

$$b = \frac{g\theta}{\Theta_0} = \phi_z$$

Almost fully compressible
if z=f(p/p0)

To solve this system with rigid lower and upper boundaries Separate variables in vertical

$$\mathbf{u}(x,y,z,t) = G(z)\hat{\mathbf{u}}(x,y,t)$$

$$\phi(x,y,z,t) = G(z)\hat{\phi}(x,y,t)$$

$$w(x,y,z,t) = H(z)\hat{w}(x,y,t)$$

$$b(x,y,z,t) = H(z)\hat{b}(x,y,t)$$

VSE and SWE result

Vertical Structure Equation

$$\frac{d^2}{dz^2}H + \lambda^2 H = 0$$

with $H(0) = H(D) = 0$

λ^2 has units of $1/g\tilde{h}$

$\tilde{h}$ is called the equivalent depth

The VSE is a Sturm - Liouville eqn

$$\mathrm{H(z)} = \mathrm{H_n}(\lambda_n z) = \sin(n\pi z/D)$$

and $\mathrm{G_n}(z) = \cos(n\pi z/D)$

Shallow Water Equations

$$\hat{\mathbf{u}}_t + U_0 \hat{\mathbf{u}}_x + f\mathbf{k}\mathbf{x}\hat{\mathbf{u}} + \nabla_h \hat{\phi} = 0$$

$$\hat{\phi}_t + U_0 \hat{\phi}_x + \Phi \nabla_h \cdot \hat{\mathbf{u}} = 0$$

Where $\Phi = g\tilde{h}$ from the VSE

Thus there are infinite number of equivalent depths $g\tilde{h}_n$

Characteristic variables

As discussed by John Thuburn

The shallow water equations with rotation

lead to a dispersion relationship:

$$\hat{\omega}(\hat{\omega}^2 - (f^2 + (k^2 + l^2)\Phi_n) = 0$$

where $\hat{\omega} \equiv \omega - kU_0$ and $\Phi_n = g\tilde{h}_n$

So there is one advective, geostrophic mode and a pair of IGWs propagating in opposite directions relative to the wind

Side bar: Spherical PE model

$$u_t + \sin\theta v - \phi_\lambda / \cos\theta = -\varepsilon(NL_u)$$

$$v_t - \sin\theta u - \phi_\theta = -\varepsilon(NL_v)$$

$$\phi_{pt} + B\omega = -\varepsilon(NL_T)$$

$$\nabla_h \cdot \mathbf{u} + \omega_p = 0$$

$$\varepsilon = Rossby\,\#, B = Burger\,\#$$

$$\text{Proper lower BC}: \ \omega(\ldots p_s) = Dp_s / Dt$$

Separation of variables works just as before.
Can be carried out for discrete system, too.

First separate variables in vertical and get vertical structures

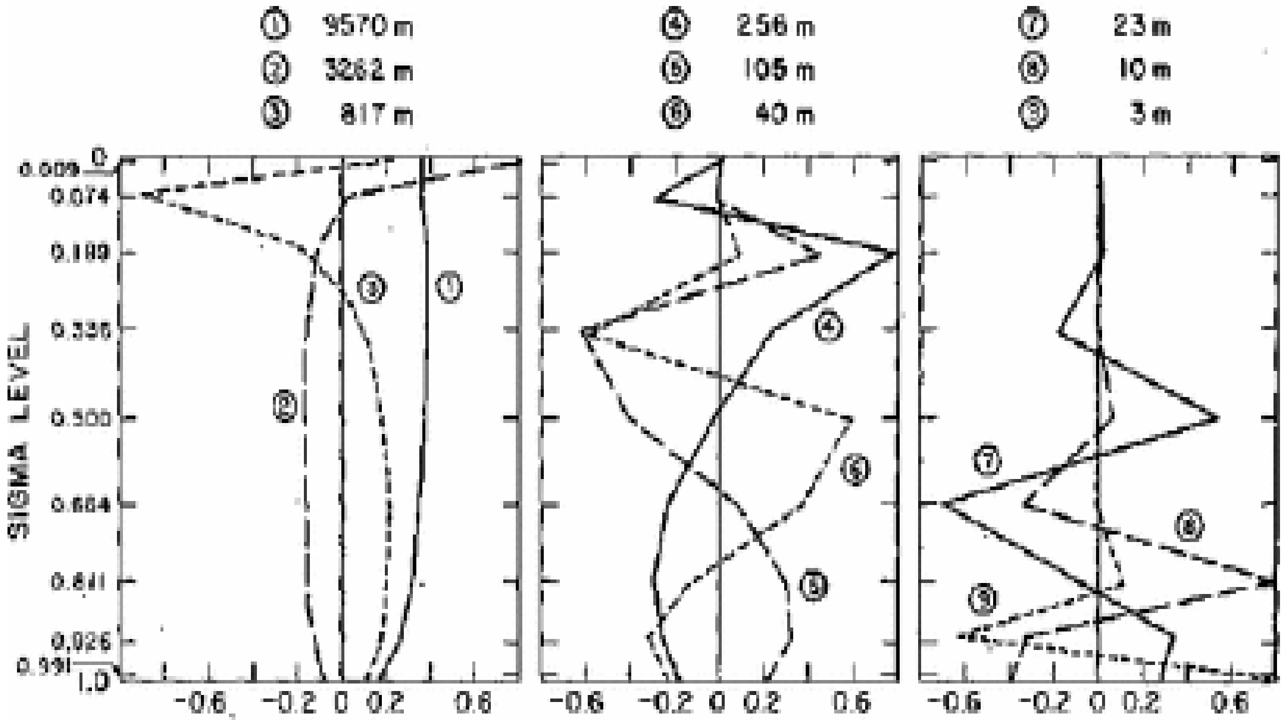


FIG. 2. Vertical profiles of eigenfunctions $\Phi_k(\hat{k})$ corresponding to equivalent heights D_k, the values of which are listed on the top.

Figure 1 : Solutions to versical structure equation (Kasahara and Puri, 1981)

Eigenvalues of VSE correspond to equivalent depths ~10km

Horizontal structure is linearized shallow water equations
Use Mercator projection and/or use Taylor series
And approximate sin(Θ)= Θ and cos(Θ)=1 (equatorial beta plane)

$$U_t - 2\Omega\varphi V + \frac{1}{a}\Phi_\lambda = 0$$

$$V_t + 2\Omega\varphi U + \frac{1}{a}\Phi_\varphi = 0$$

$$\Phi_t + \frac{gH}{a}(U_\lambda + V_\varphi) = 0.$$

$$\begin{bmatrix} U(\lambda,\sigma,t) \\ V(\lambda,\sigma,t) \\ \Phi(\lambda,\varphi,t) \end{bmatrix} = \begin{bmatrix} \widehat{U}(\varphi) \\ i\widehat{V}(\varphi) \\ \widehat{\Phi}(\varphi) \end{bmatrix} e^{i(s\lambda - \sigma t)}$$ which results in:

$$-\sigma\widehat{U} - 2\Omega\varphi\widehat{V} + \frac{s}{a}\widehat{\Phi} = 0$$

$$-\sigma\widehat{V} - 2\Omega\varphi\widehat{U} + \frac{1}{a}D\widehat{\Phi} = 0$$

$$-\sigma\widehat{\Phi} + \frac{gH}{a}(s\widehat{U} + D\widehat{V}) = 0 \text{ , with } D \equiv \frac{d}{d\varphi}.$$

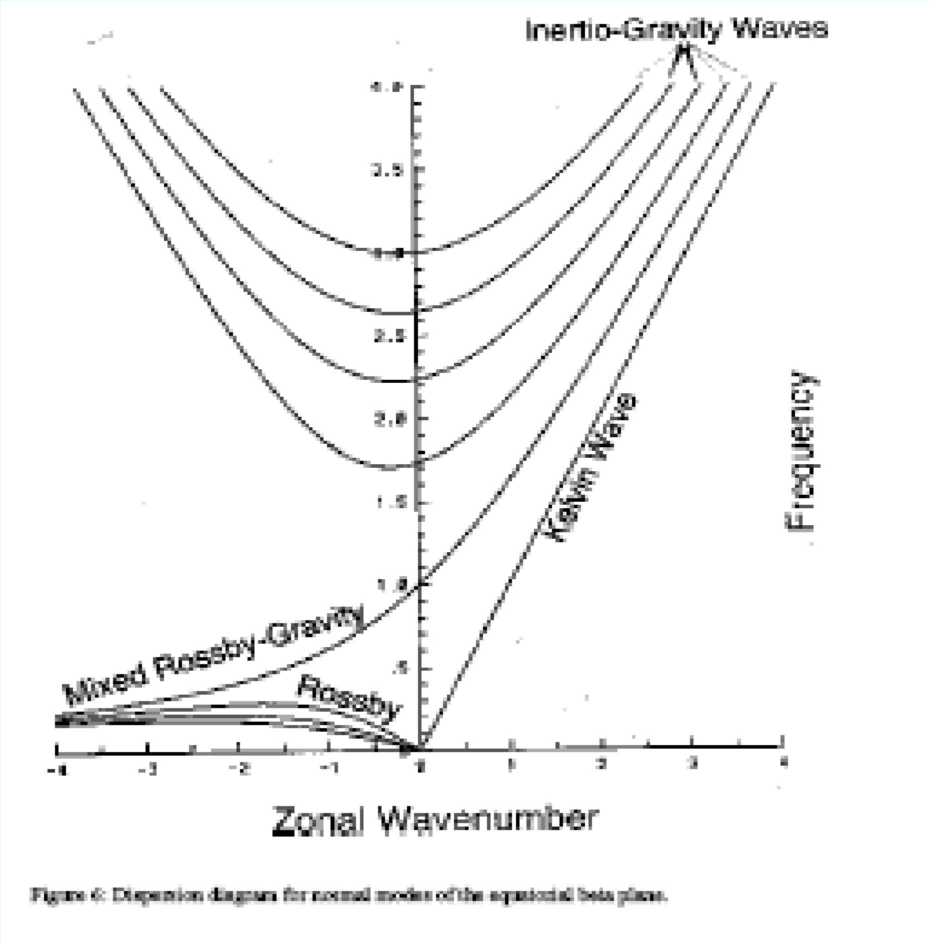
This can be reduced to a single ordinary equation for $\widehat{V}$:

$$D^2\widehat{V} + (\varepsilon\sigma^2 - s^2 + \frac{s}{\sigma} - \varepsilon\varphi^2)\widehat{V} = 0 \text{ , with } \varepsilon \equiv \frac{4\Omega^2 a^2}{gH},$$ which

is often referred to as Lamb's parameter. The above equation is identical in

Solution in terms of parabolic cylinder functions

Spectrum of wave speeds separates into slow and fast Gravity and Rossby



Fastest waves (10km eq depth) have phase speeds of 300m/s

Characteristic variables (cont)

To study the IBVP it is simpler to consider propagation in the x direction only .

So we ignore partial derivatives with respect to y (i.e. set l=0)

It is useful to define the following combination of variables for each λ_n :

$$\xi_n \equiv u_n + \lambda_n \phi_n, \quad \eta_n \equiv u_n - \lambda_n \phi_n, \quad c_n \equiv 1/\lambda_n = \sqrt{g\tilde{h}_n}$$

The SWE for each n becomes:

$$\xi_{nt} + (U_0 + c_n)\xi_{nx} - f v_n = 0$$

$$\eta_{nt} + (U_0 - c_n)\eta_{nx} = 0$$

$$v_{nt} + U_0 v_{nx} + \tfrac{1}{2} f(\xi_n + \eta_n) = 0$$

Return to Characteristic variables (cont)

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In GREAT shape for IBVP –have characteristic variables for each n!

BUT

The for each n part means in order to compute CVs one must perform a projection onto vertical structure functions

We're talking about GLOBAL models-Why worry about this?

- Local mesh refinement just like IBVP
- Need to determine incoming and outgoing characteristics or the IBVP will be 'ill-posed' at horizontal boundaries
- 10km eq. depth vertical mode IGW (Lamb wave) always incoming and outgoing for Earthlike mean velocities (c=300m/s)
- Internal IGWs usually have their direction determined by the mean flow

Example: specify variables on inflow (mean wind is subcritical)

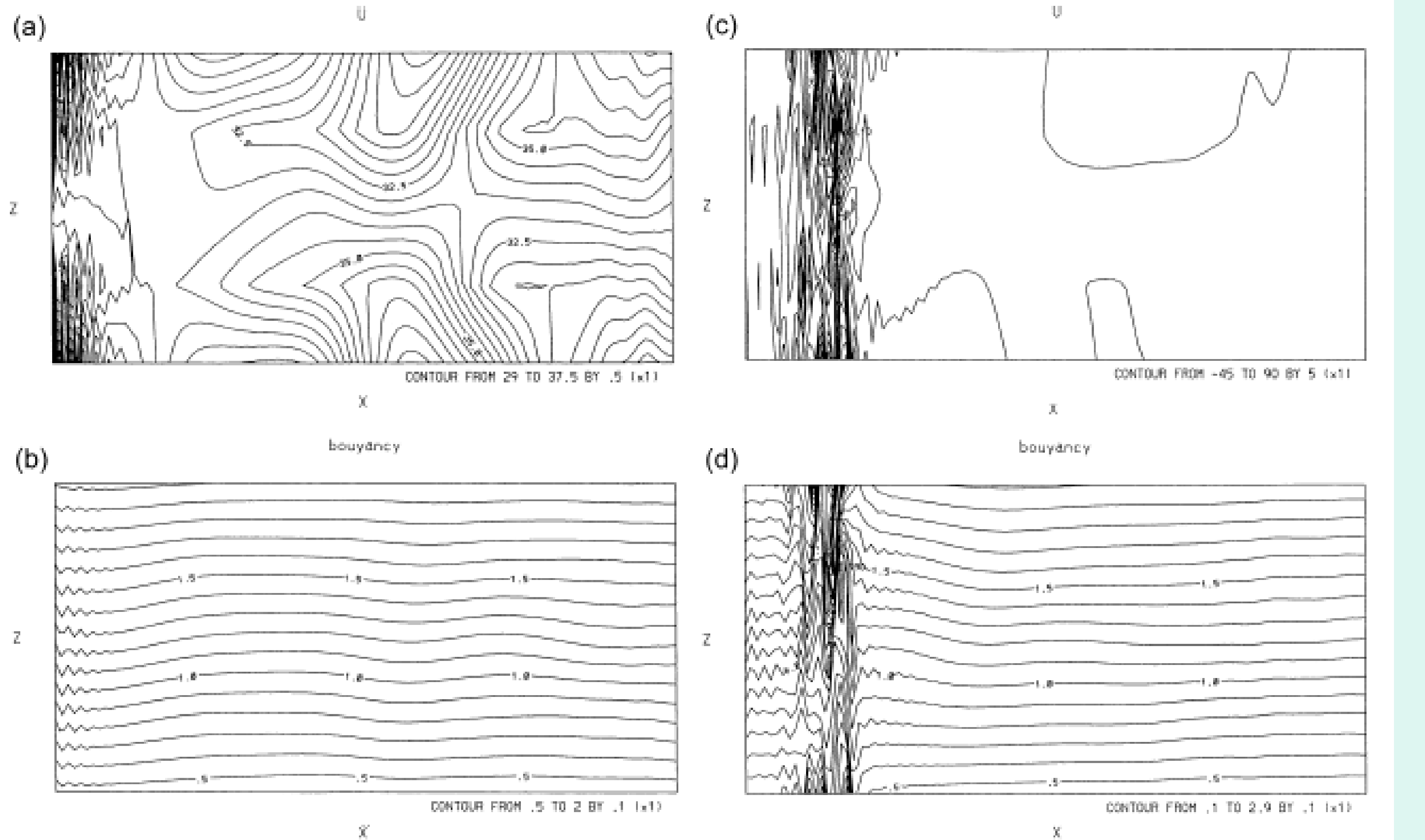


FIG. 3. (a) Horizontal velocity, U, at t = 28 h for the traditional hydrostatic limited-area model ($\delta = 0$) with subcritical U_0. (b) As in (a) but for the buoyancy, b. (c) As in (a) but at t = 56 h. (d) As in (b) but at t = 56 h.

Solution: add dissipation

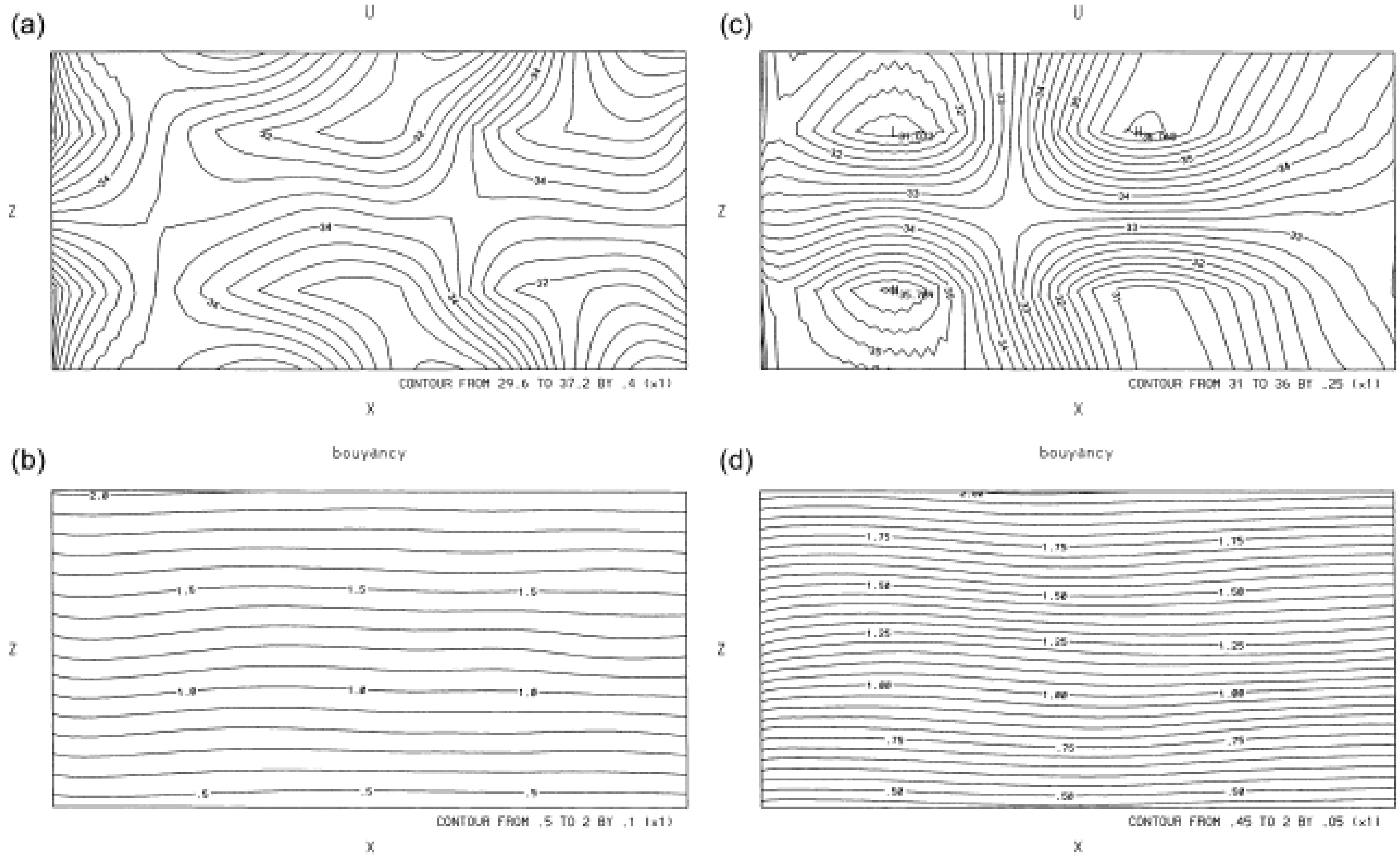


FIG. 4. (a) Horizontal velocity, U, at t = 28 h for the modified hydrostatic limited-area model (δ = 0.3) with subcritical U_s. (b) As in (a) but for the buoyancy, b. (c) As in (a) but at t = 56 h. (d) As in (b) but at t = 56 h.

Main points

- Hydrostatic system used/useful: filters vertically propagating (internal) acoustic waves. CFL not a killer.
- System can be reduced to SWE and written in characteristic form
- Price to be paid for IBVP (vertical coupling of characteristics non-locality)
- Makes life difficult for local and adaptive mesh refinement

Asymptotic Balance

Equations in normal mode form

$$x' = \varepsilon(N(x,y) - i\sigma x)$$
$$y' + i\sigma y = \varepsilon M(x,y)$$

Amplitude of fast waves small but slaved component important

Approximate slave relation: slow manifold

$$y_b \approx \varepsilon M(x,0)/(i\sigma)$$
$$x' = \varepsilon(N(x,0) - i\sigma x)$$

Starting with PE (on a beta plane) this give QG

Balance and turbulence

$$q_t + J(\psi, q) = 0,$$

$$q = \nabla^2\psi + (f^2 / N^2)\psi_{zz} \equiv L\psi$$

Fjortoft Constraints

$$\text{Total Energy } E = -\int (\psi L\psi) dV$$

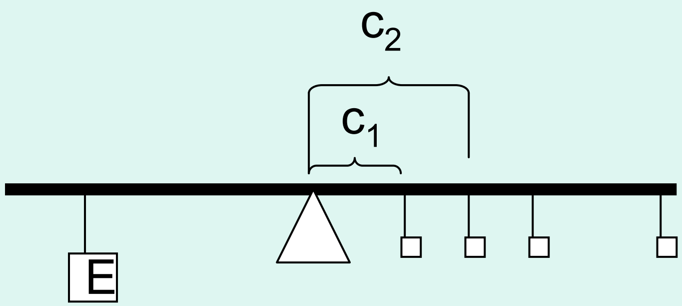
$$\text{Pot. Enstrophy } EN = \int (L\psi)^2 dV$$

$$L\varphi_n = -c_n^{\ 2}\varphi_n$$

$$\psi = \sum a_n \varphi_n, \ E = \sum |a_n|^2 c_n^{\ 2}$$

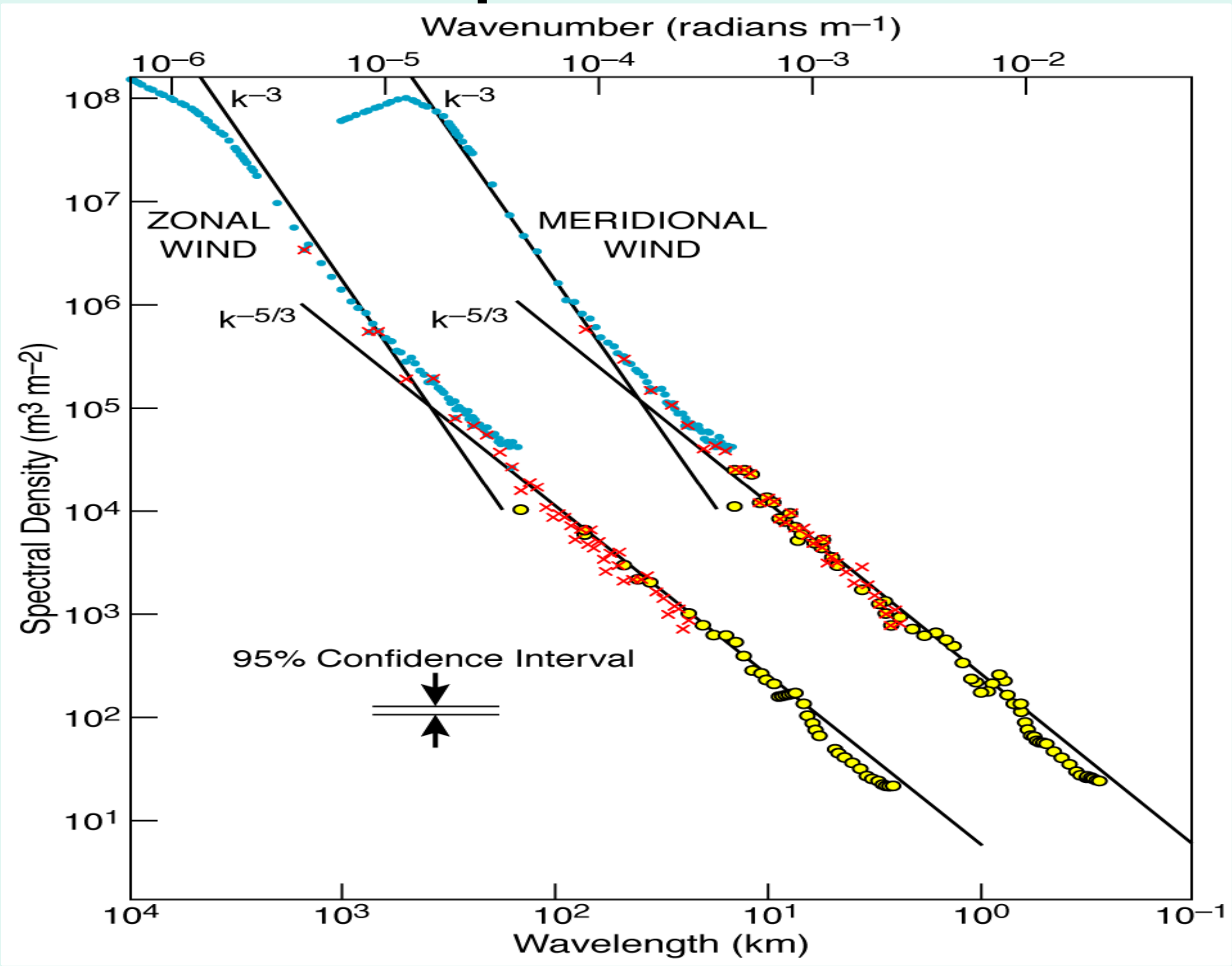
$$\text{Let } e_n \equiv |a_n|^2 c_n^{\ 2}$$

$$E = \sum e_n \ \text{ and } EN = \sum c_n^{\ 2} e_n$$

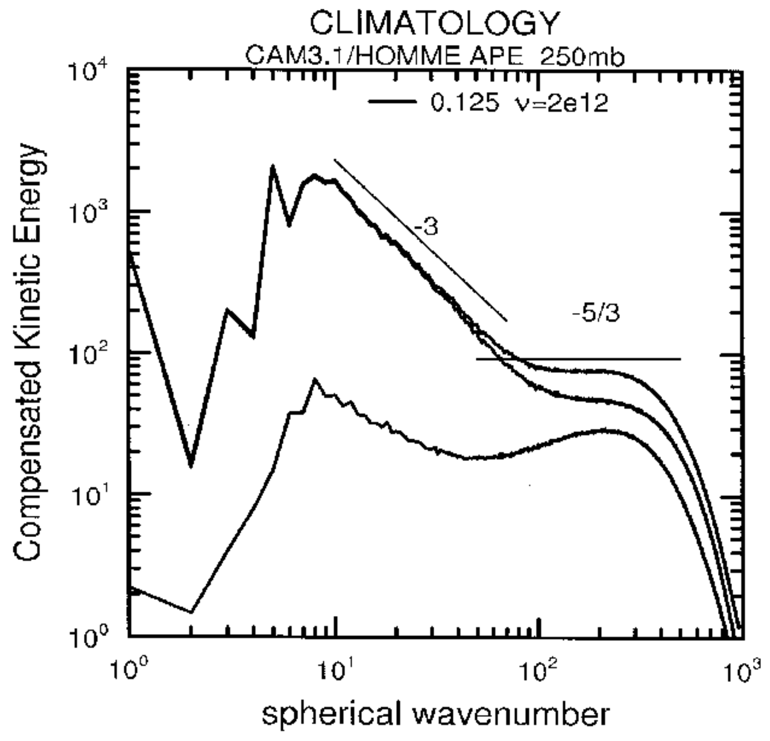


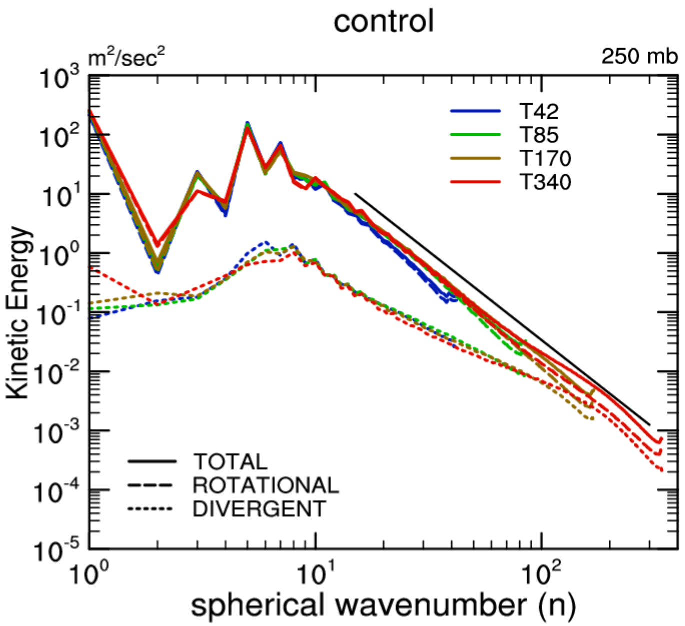
Cannot maintain balance if
Energy moves toward large n
Power law spectrum $k^{-5/3}$ and k^{-3}

The Nastrom & Gage Spectrum



HOMME and Spectral results (Taylor and Williamson)





Back to dissipation

- Atmosphere dissipates by cascading energy to molecular scales
- Finite resolutions means cascade tail must be parameterized
- Need eddy viscosity to prevent equipartition (k^{-2})
- 2D/QG turbulence (k^{-3}) implies constant eddy turnover time as k increases
- Eddy viscosity often hyperviscosity with almost fixed e-folding time at truncation limit

Short list of references

- Characteristics: *Partial Differential Equations of Applied Mathematics* by E Zauderer, Wiley 1983
- ‘Ill-posedness’ of hydrostatic primitive equations: Temam and Tribbia *JAS* 2003 and Rosseau, Temam and Tribbia *Discrete Cont. Dyn. Syst.* 2005
- Turbulence: *Atmospheric and Oceanic Fluid Dynamics* by G. K. Vallis Cambridge University Press 2007