Introduction to Voronoi Diagrams and Delaunay Triangulations

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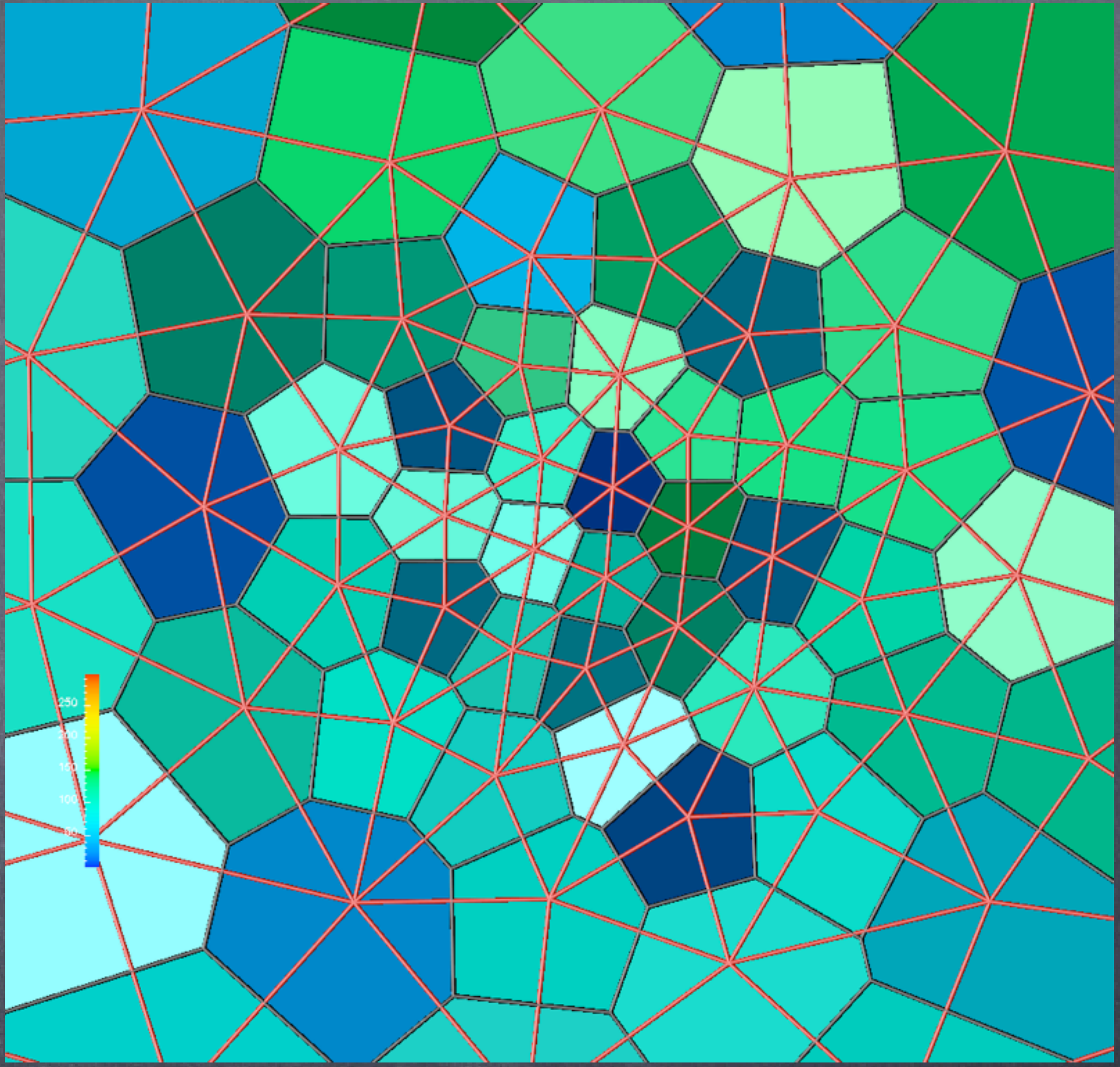
Outline

Define Voronoi diagrams, define Delaunay triangulations and discuss how the two are related.

Introduce and define Centroidal Voronoi diagrams.

Show some variable resolution meshes applicable to climate system modeling.

Show some results using variable resolution meshes to highlight the challenges ahead.



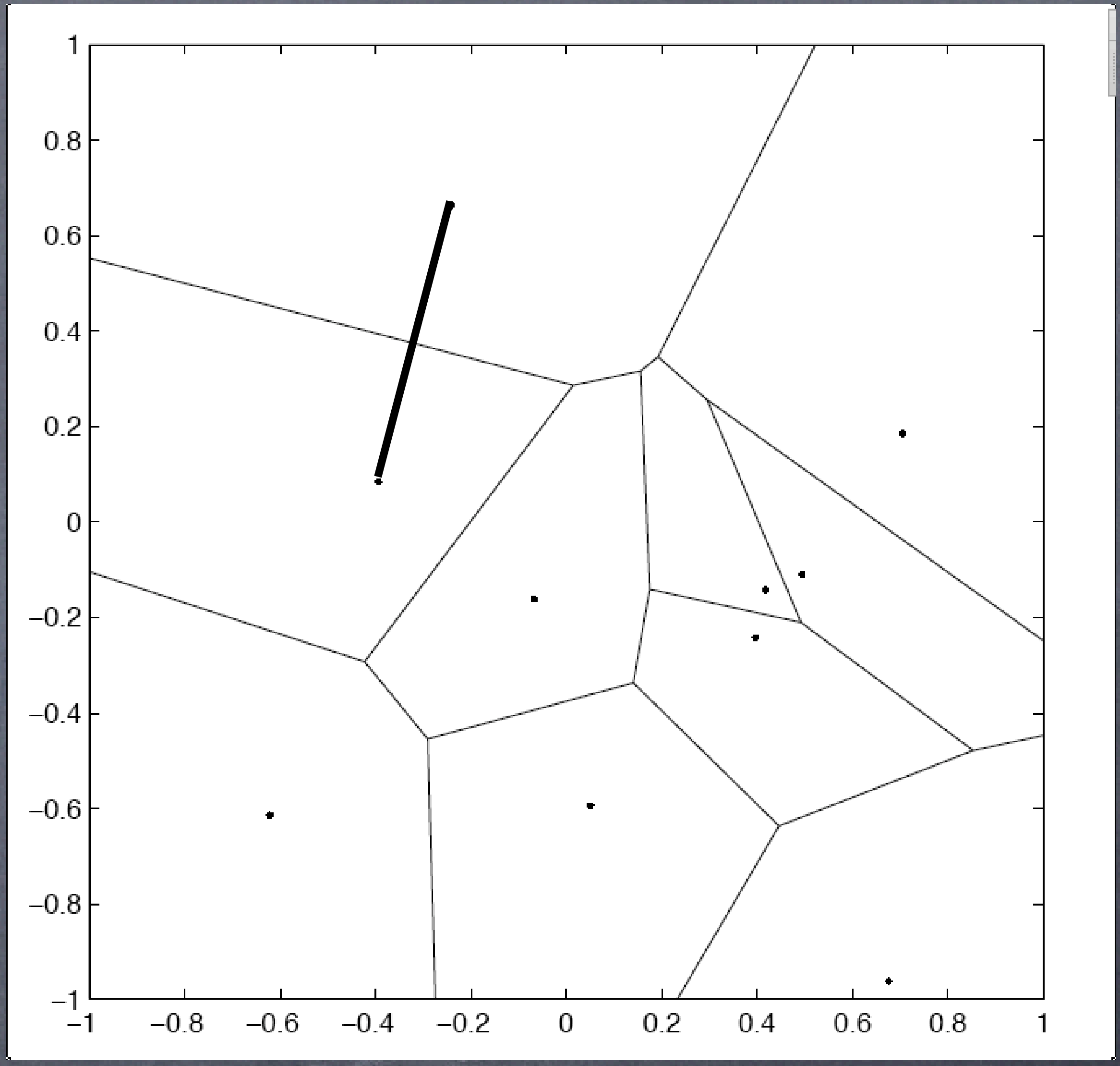
Voronoi Diagram (colored)
and the dual Delaunay triangulation

Definition of a Voronoi Tessellations

Given a region, S
And a set of generators, z_i ...

The Voronoi region, V_i, for each z_i is the set of all points closer to z_i than z_j for j not equal to i.

We are guaranteed that the line connecting generators is orthogonal to the shared edge and is bisected by that edge.



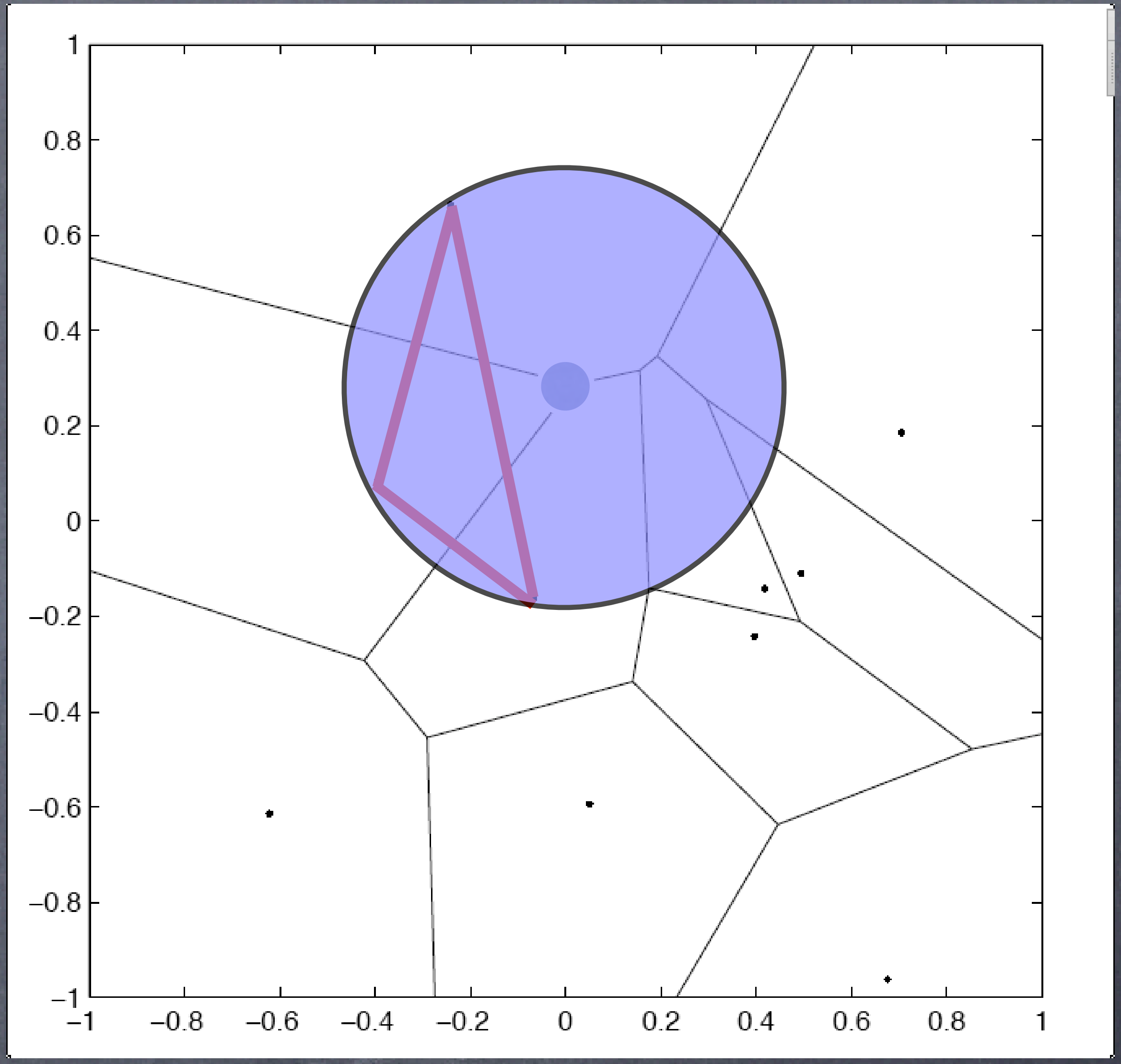
Note: This Voronoi tessellation is for illustration only -- it is not intended to be a quality grid.

Definition of a Delaunay Triangulation

For each Voronoi diagram there is a unique Delaunay triangulation, and vice versa.

The nodes (or generators) of the Voronoi diagram are the vertices of the Delaunay triangulation.

Each triangle of the Delaunay triangulation is associated with one vertex of the Voronoi diagram.



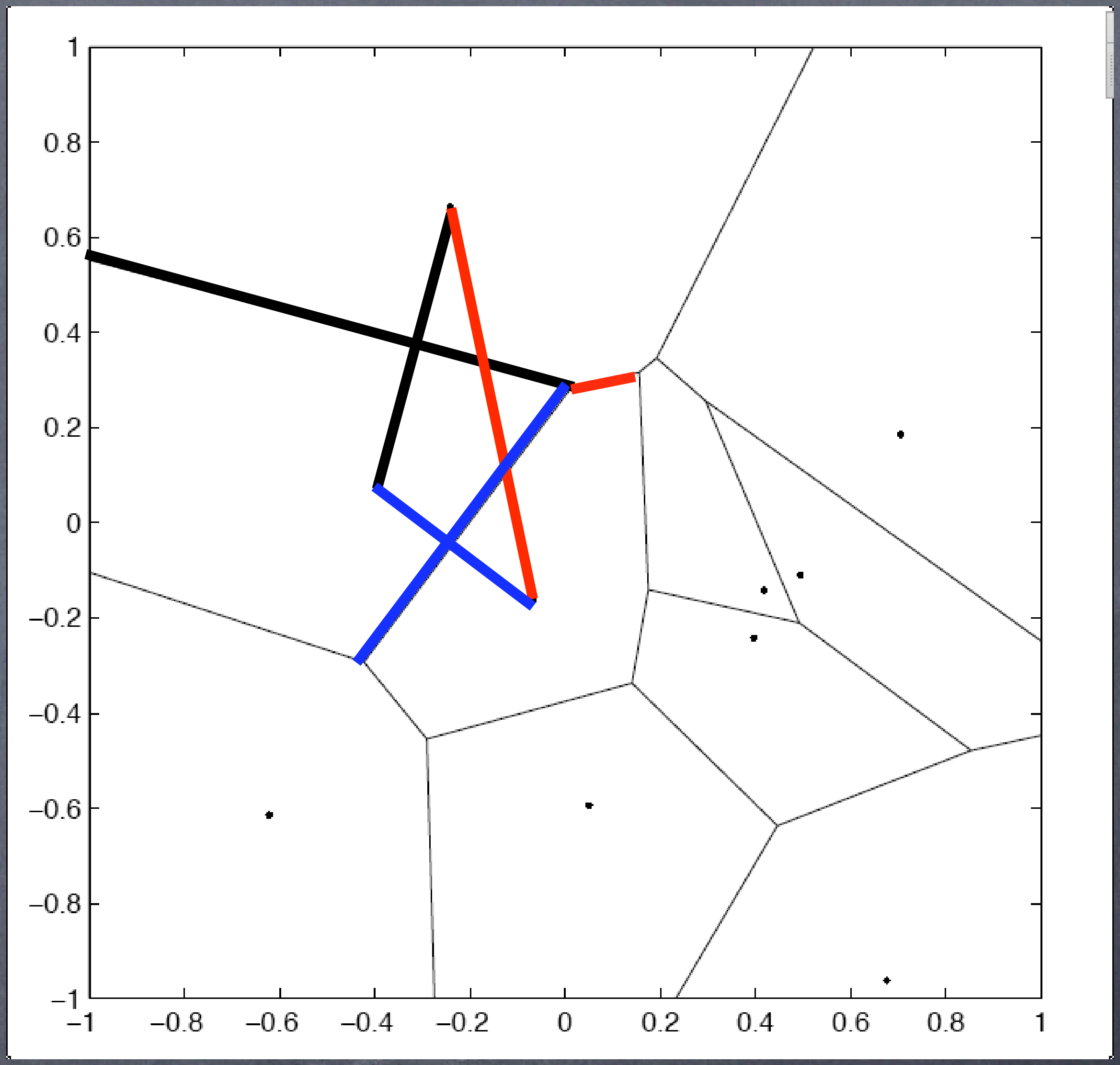
The vertex is located at the center of the circumscribed circle.

Relationship between VD and DT edges

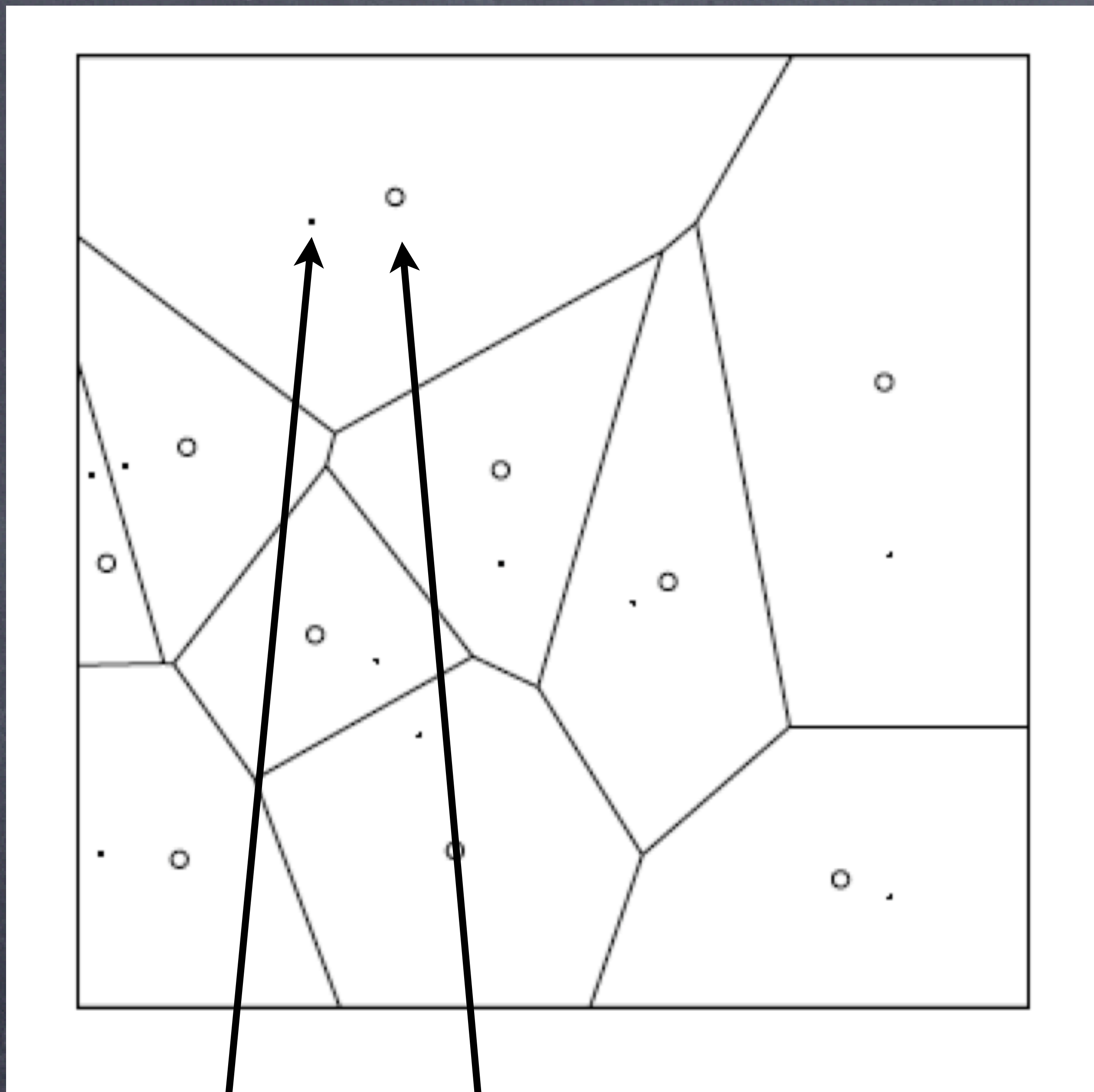
Each cell edge of the Voronoi diagram is uniquely associated with one cell edge of the Delaunay triangulation.

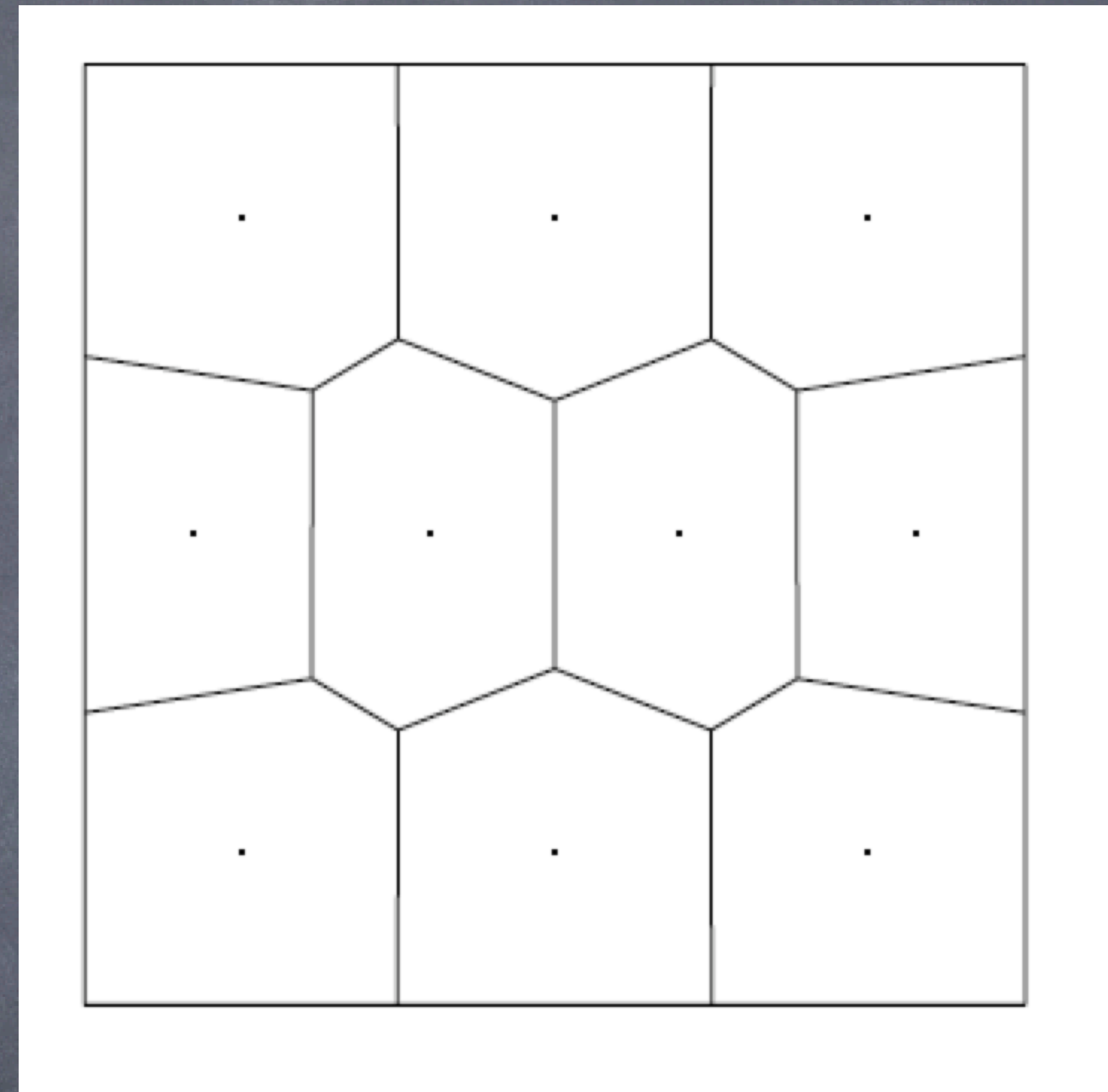
Each pair of edges are orthogonal, (but do not necessarily intersect!)

If the pair of edges do intersect (or if the lines segments are extended to point where they intersect), then the intersection point will bisect the line segment connecting generators.



Definition of a Centroidal Voronoi Tessellations





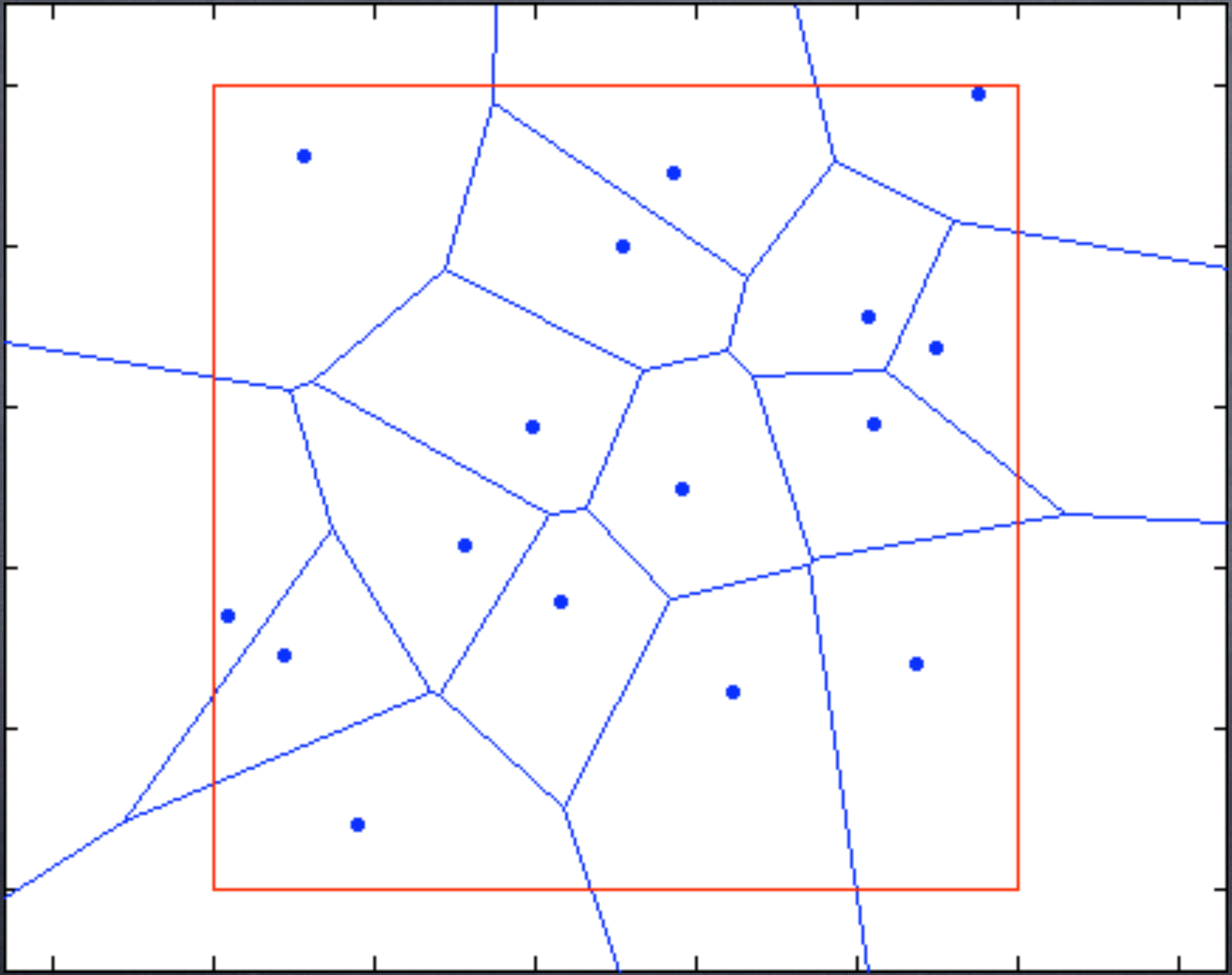
z_i

z_i^* = center of mass wrt a user-defined density function

$$z^* = \frac{\int_V w\rho(w)\,dw}{\int_V \rho(w)\,dw}$$

Iterating toward and CVT ....

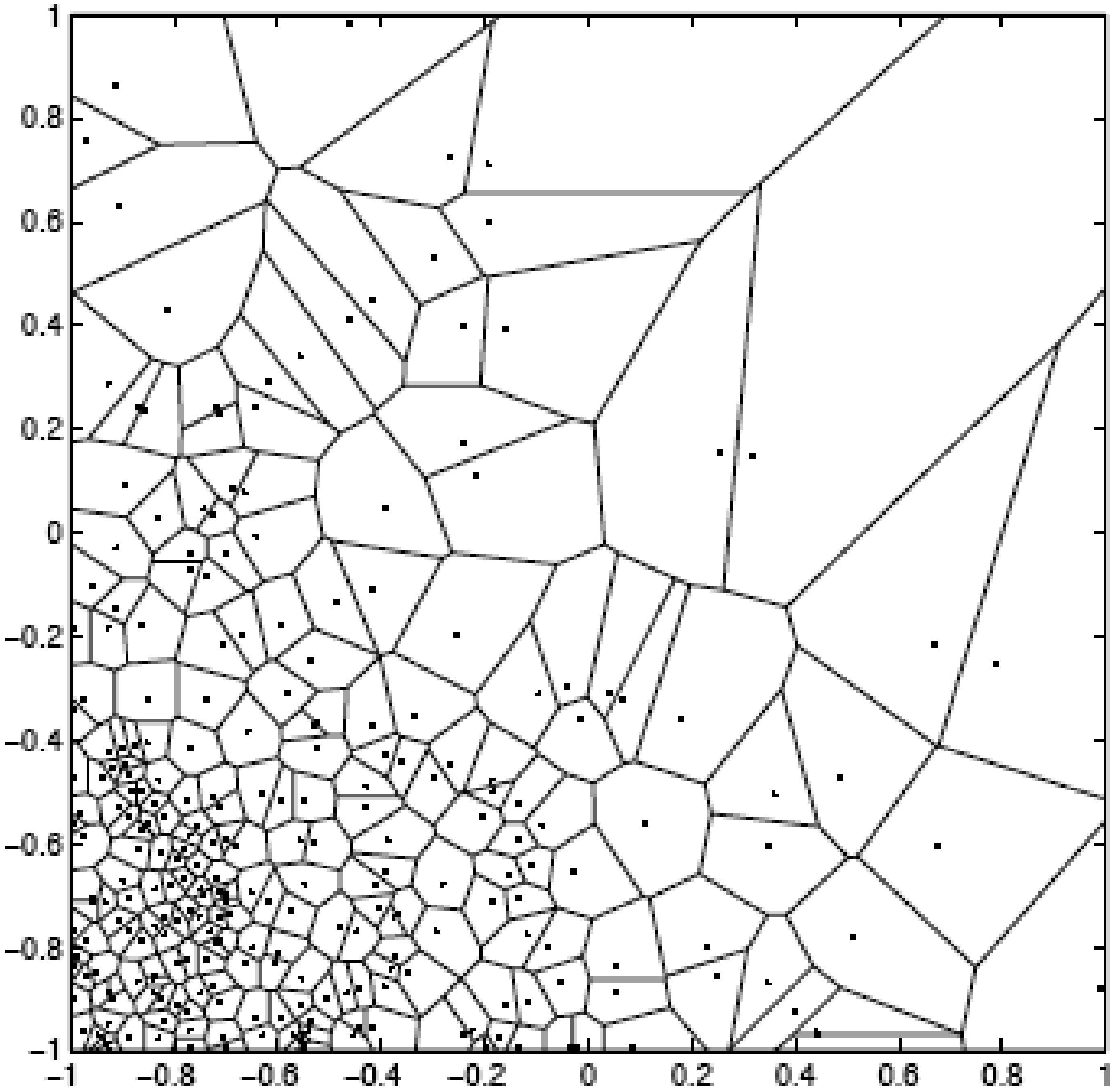
density is uniform, so generators migrate toward the geometric center.



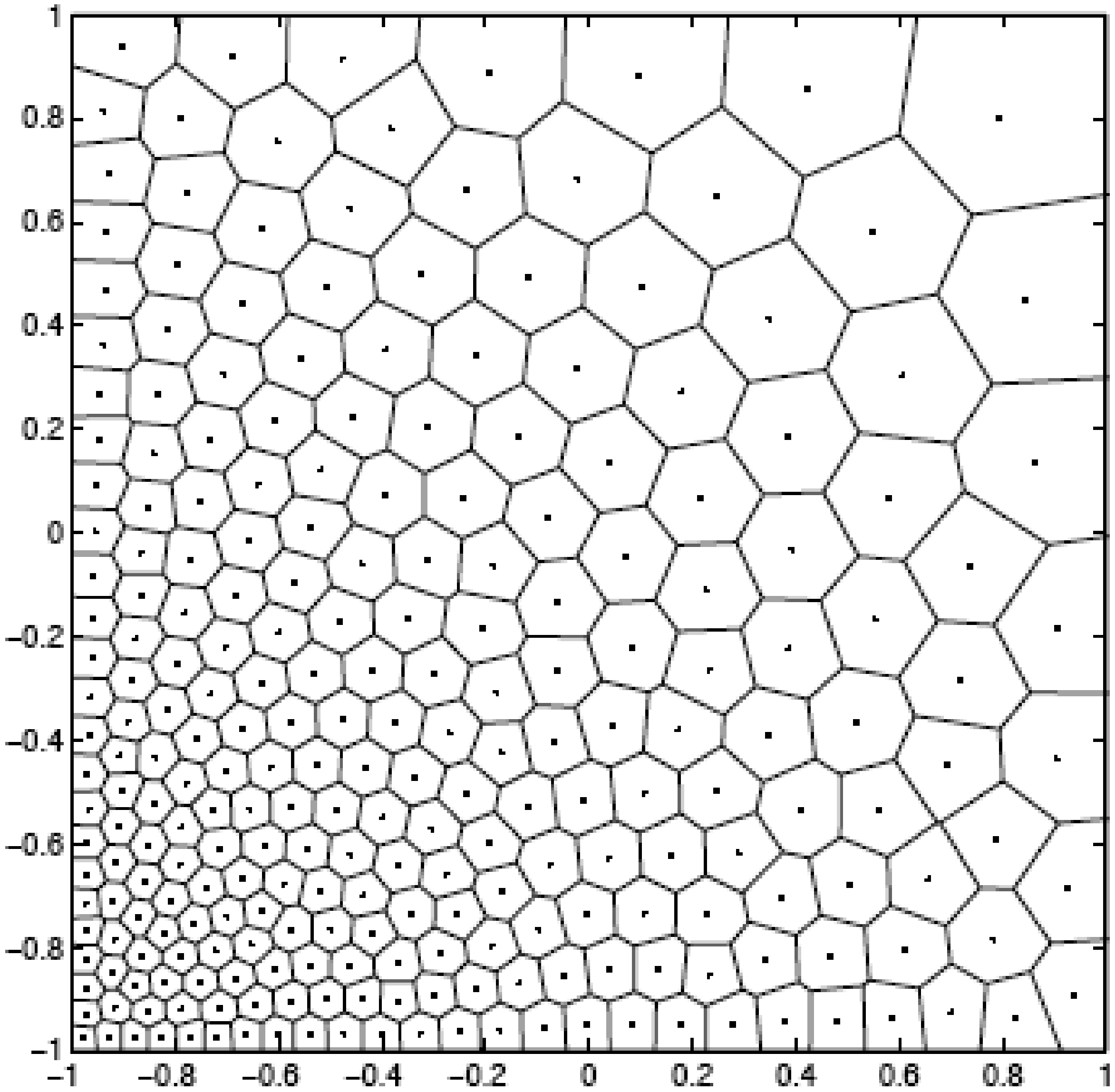
Non-uniform Centroidal Voronoi Tessellations

Distribute generators in such a way as to make the grid regular.

Also biases the location of those generators to regions of high density.



Random sampling



Centroidal Voronoi

Methods for determining the CVT.

Lloyd's Algorithm

1) Given a set of generators, draw Voronoi diagram
2) Find center of mass of each cell (via numerical integration)
3) Measure error (distance between generators and center of mass)
4) Move generators to center of mass
5) Error too big? If yes, to go 1).

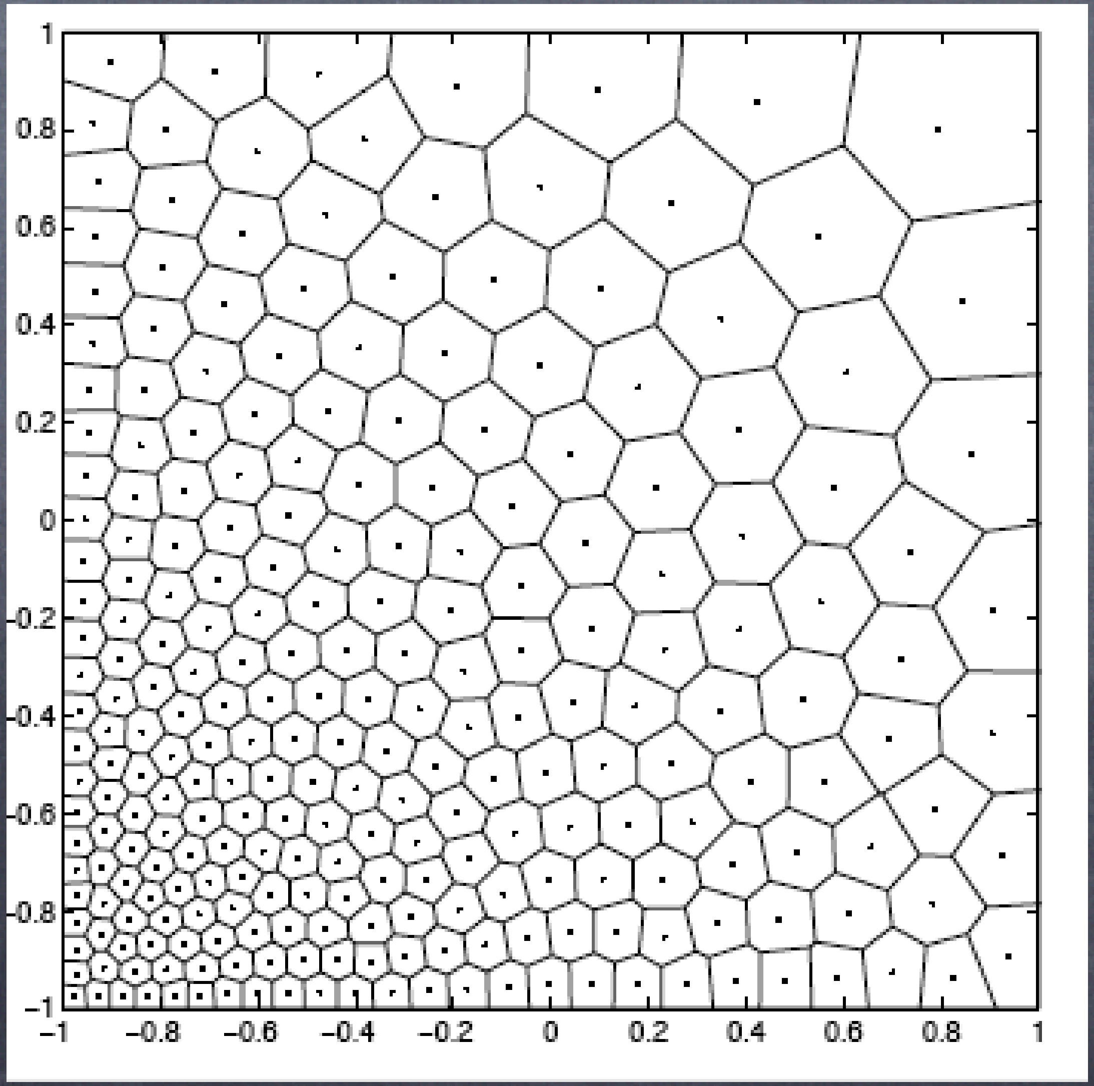
Statistical Sampling Algorithm

1) Randomly sample a point, X, in the domain.
2) Generate a random number, R, between rho(min) and rho(max)
3) Discard point if R<rho(X)
4) Assign sample point to closest generator
5) Return to 1) until N points have been retained
6) Move generator to arithmetic mean of associated sample points.
7) Measure error (distance of generators movement)
6) Error too big? If yes, to go 1).

CVTs and their counterparts on the sphere (SCVTs) have their roots in applied math ...

Gersho's conjecture

Pick some CVT density field (with minimal constraints on smoothness). Gersho's conjecture (now proven in 2D) tells us that as we add generators all of the cells evolve toward perfect hexagons. By extension, the dual Delaunay triangulation evolves toward equilateral triangles.

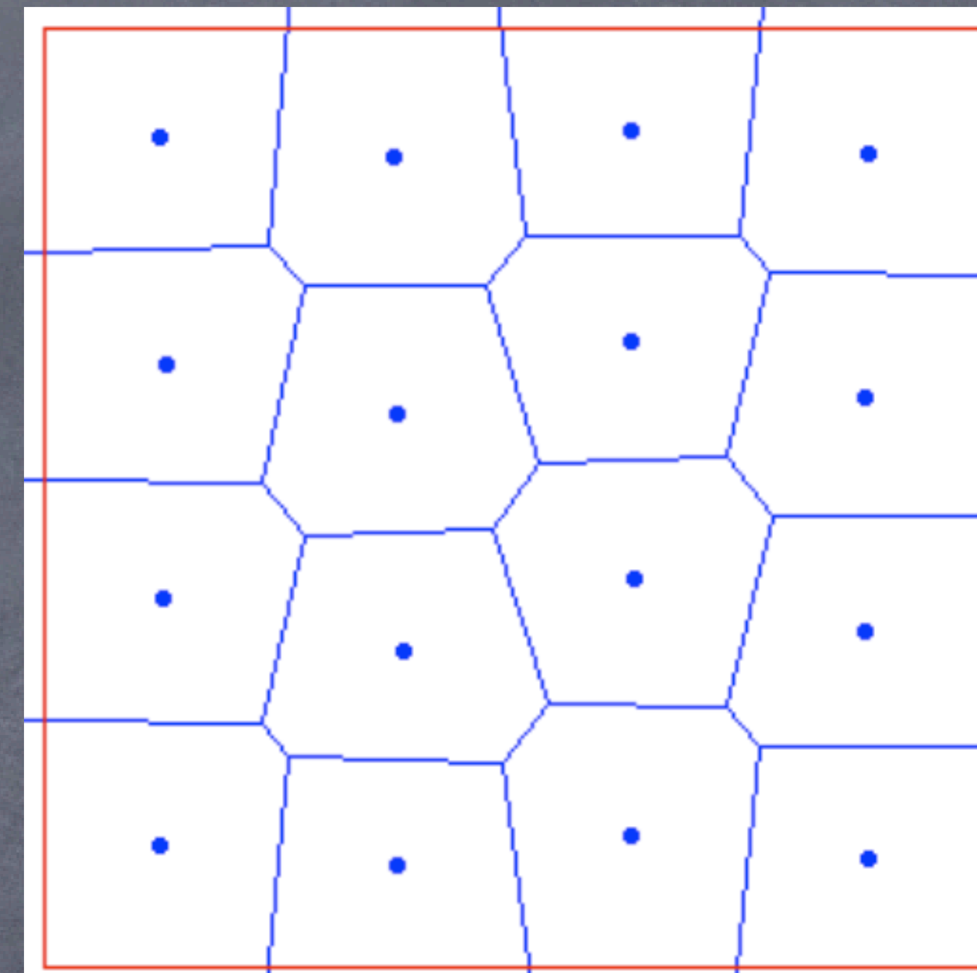


Optimal Sampling (1/2)

Domain R

Task: Based on N samples, obtain an optimal estimate of precipitation in R.

N buckets

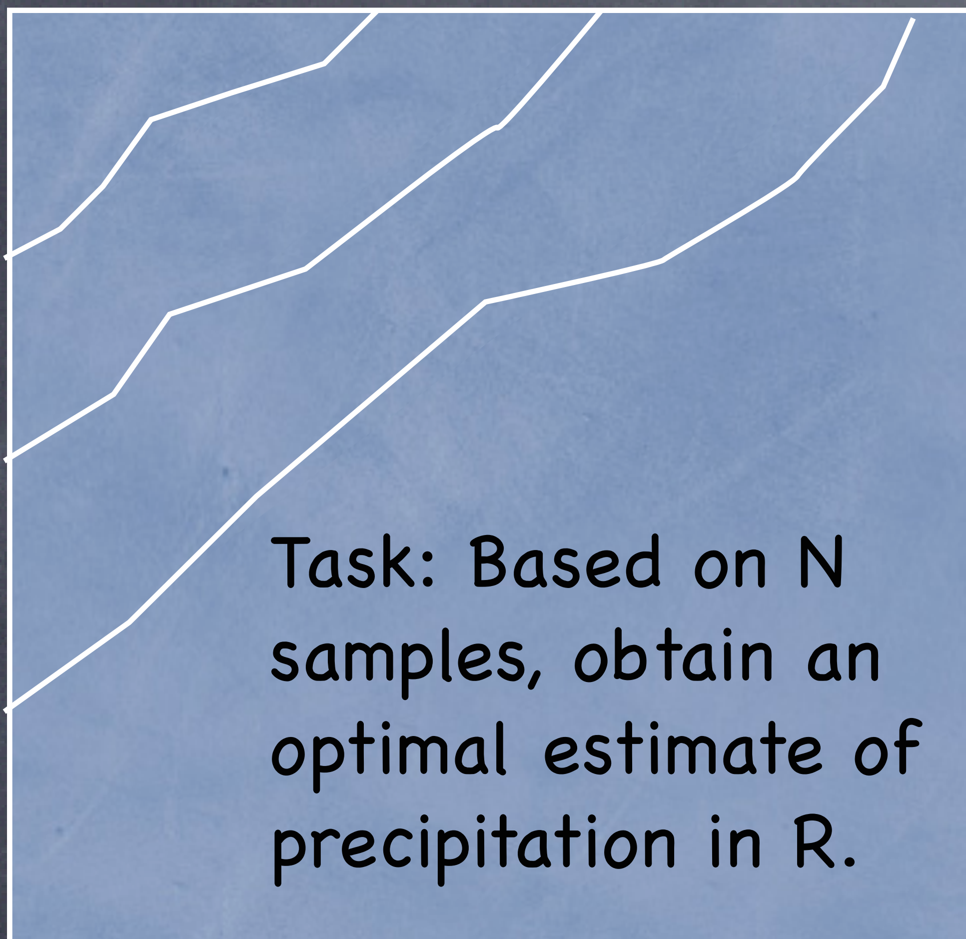


Problem #1: Given N buckets and no prior knowledge of precipitation (P) in R, where do you sample precipitation?

Answer #1: The buckets are spaced "uniformly thoughout the domain" as a centroidal Voronoi diagram based on a constant density.

Optimal Sampling (2/2)

Domain R



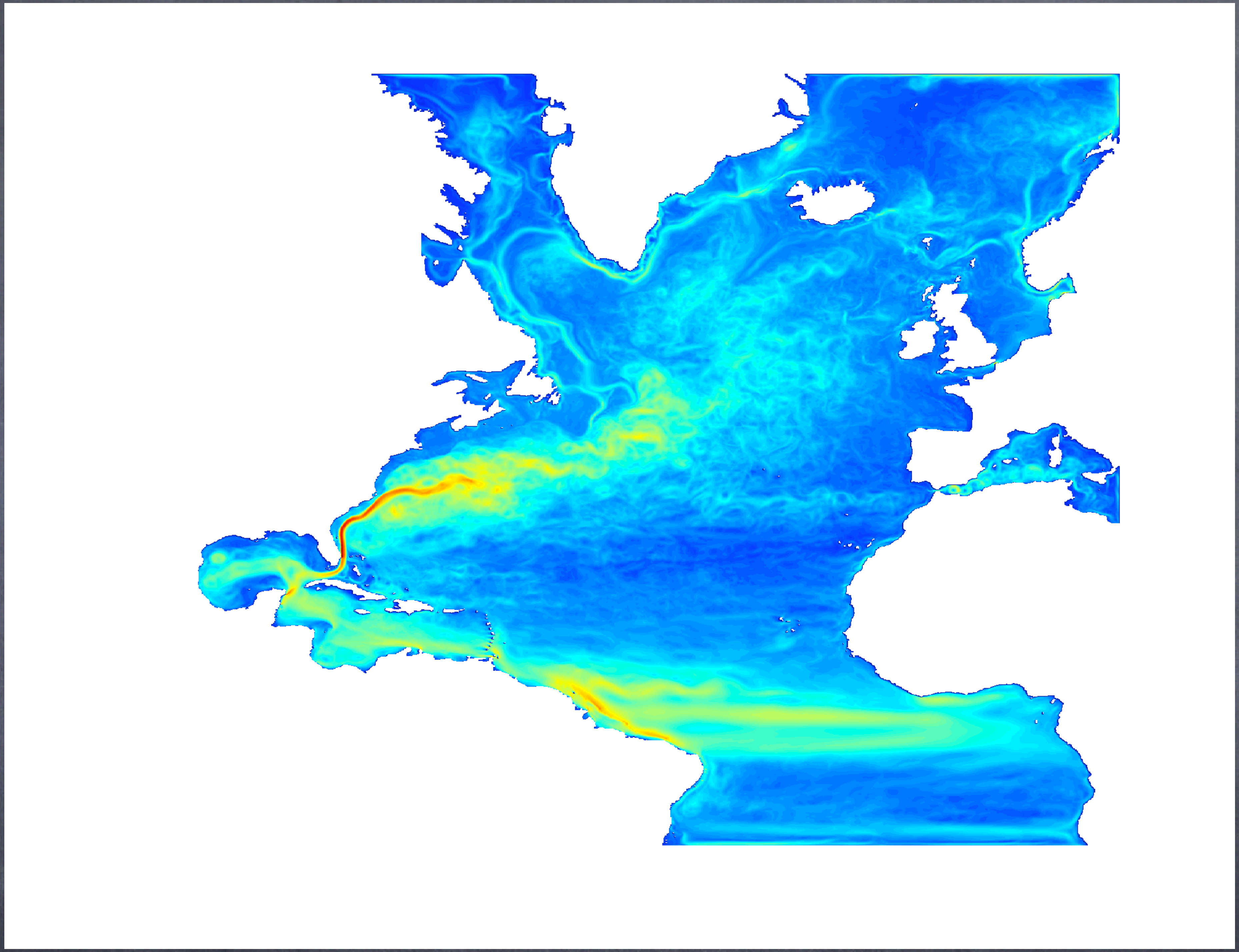
N buckets

my apologies,
Voronoi Diagram to follow ...

Problem #2: Given N buckets and **an estimate** of precipitation (P) in R, where do you sample precipitation?

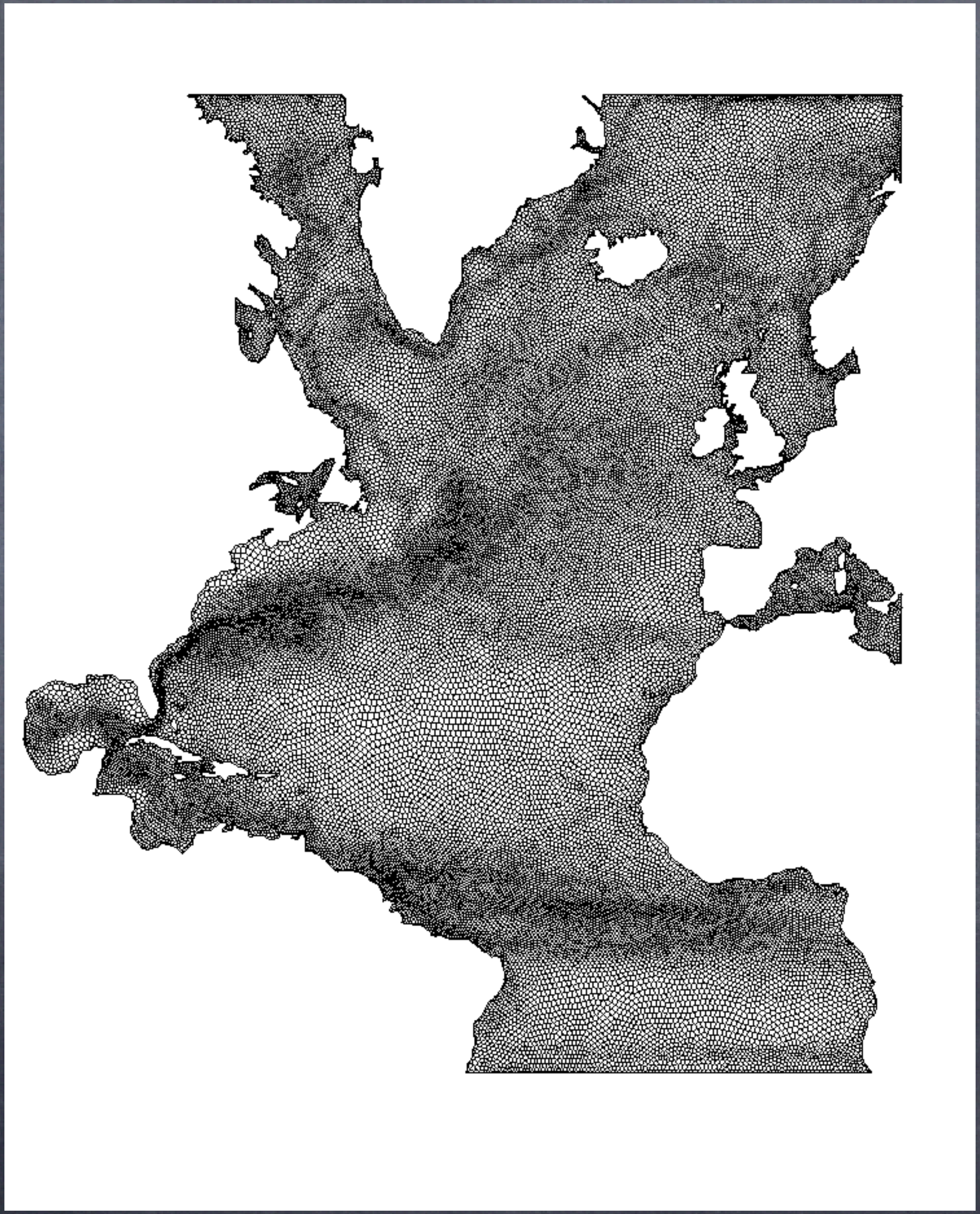
Answer #2: The buckets are spaced as a centroidal Voronoi diagram based on a density field rho=sqrt(P).

Warning: Some (non-atmosphere) applications to follow ....

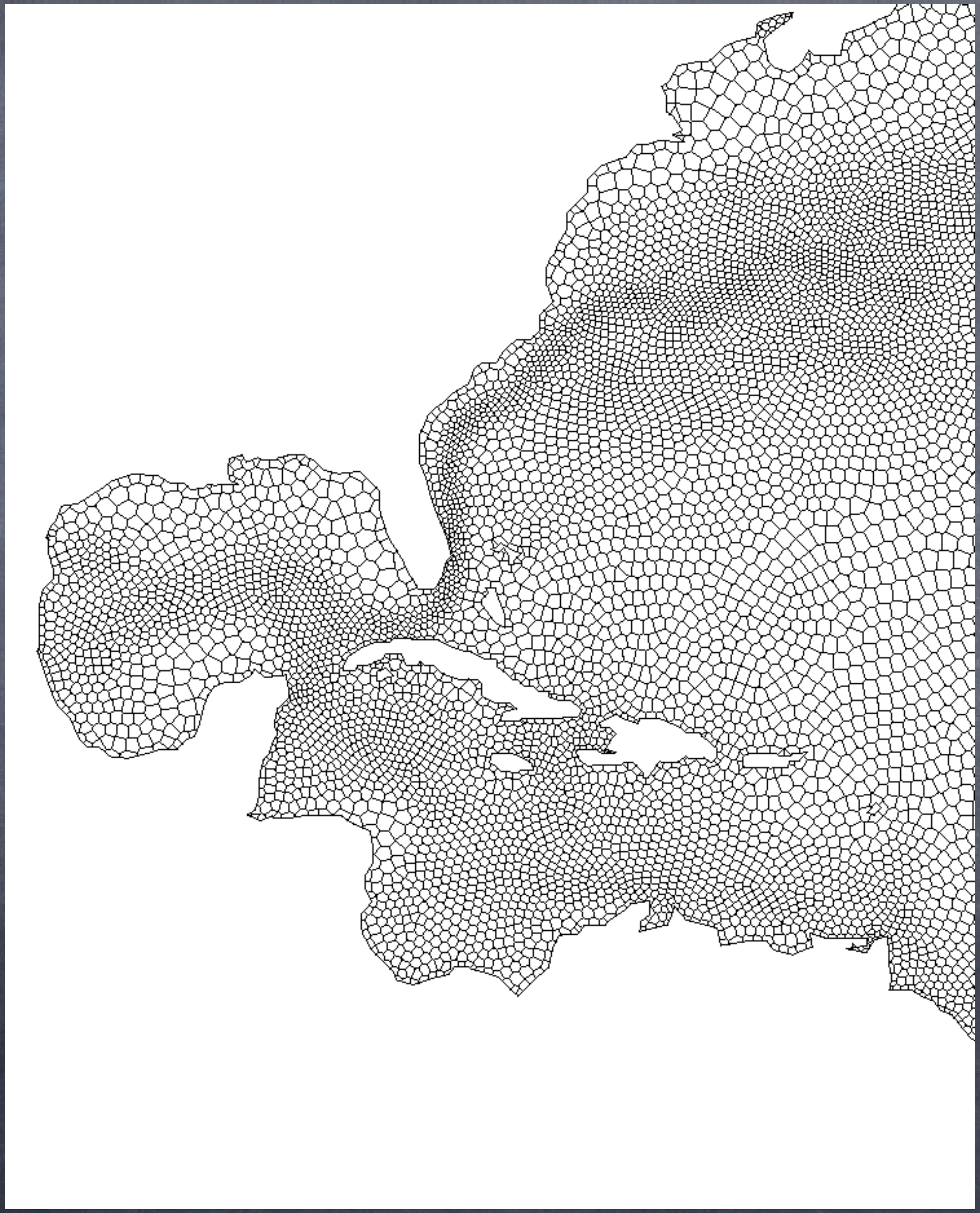


Kinetic energy from an eddy-resolving ocean simulation.

Resulting Centroidal Voronoi Diagram ...



Resulting Centroidal Voronoi Diagram, a closer look at the Gulf Stream region.

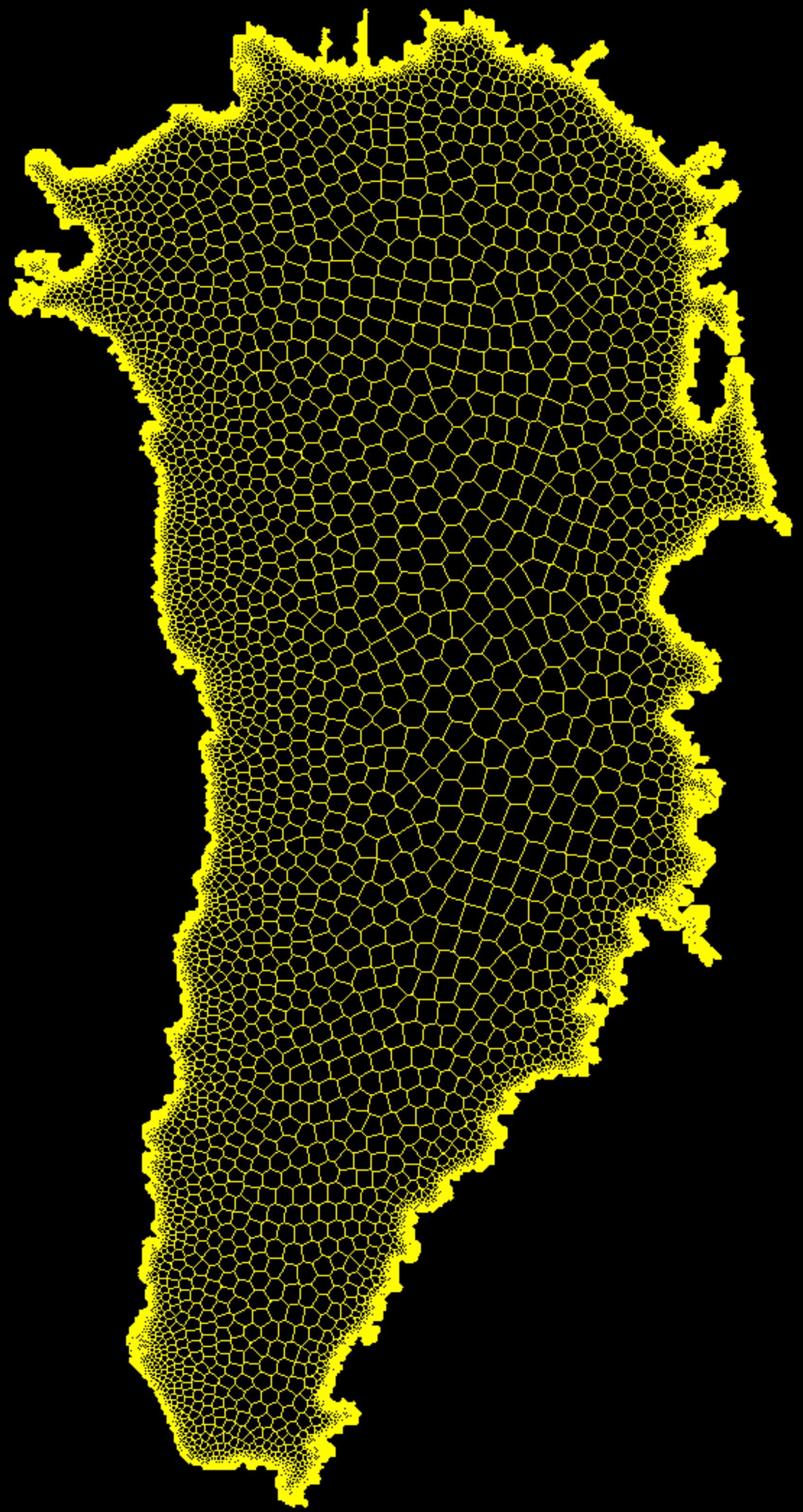


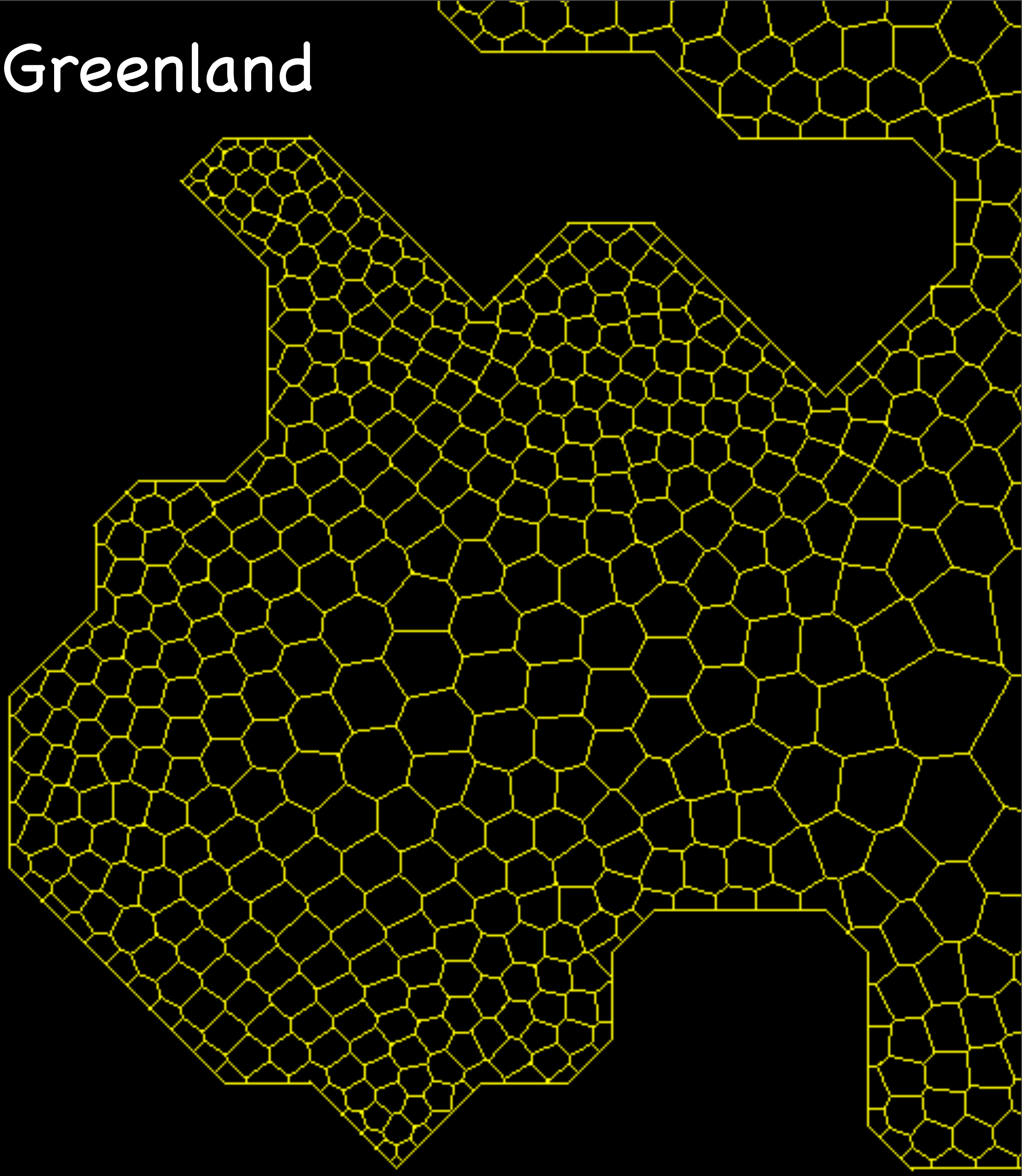
And adding more nodes ...

The result is that we can generate meshes with "eddy-resolving" resolution (~10 km) in localized regions of intense eddy activity using ~1/10 the degrees of freedom required for a uniform 10 km mesh.

The (yet unrealized) potential is that IPCC-class ocean simulations can included eddies (in specific regions) within 10 years.

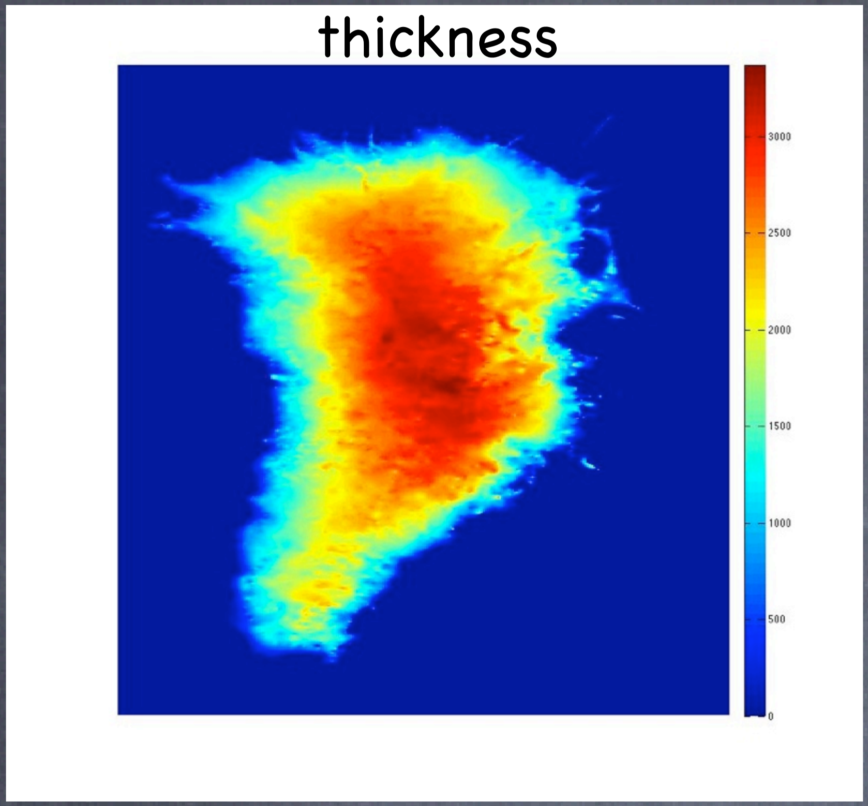
Example Grids: Greenland





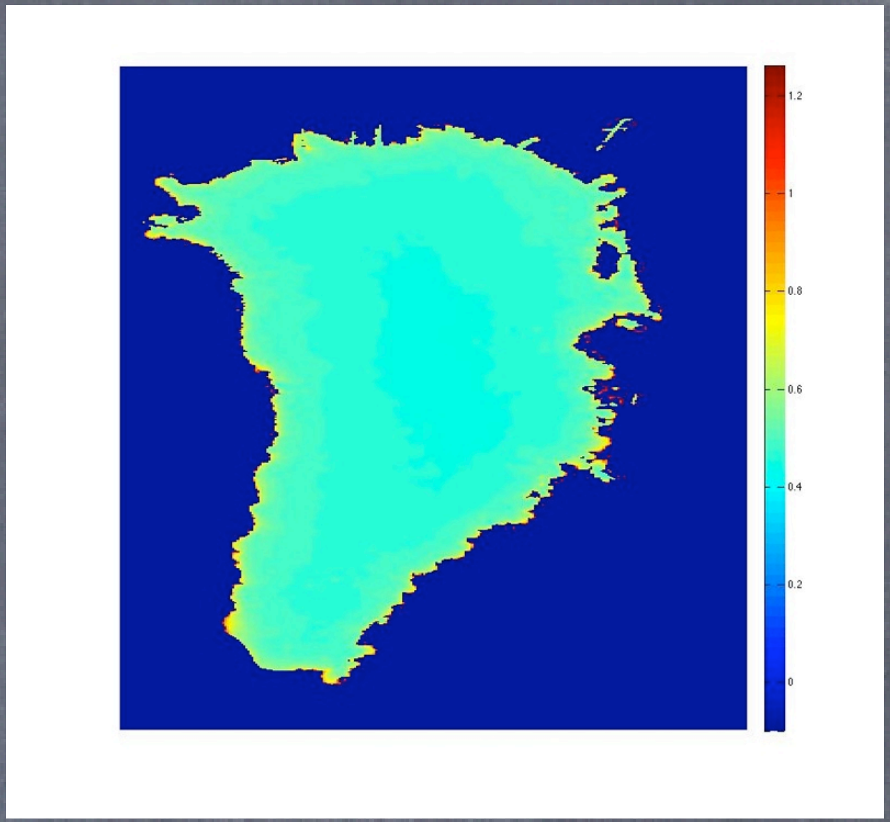
Example Grids: Greenland: How we developed these grids.

Bamber 5 km dataset



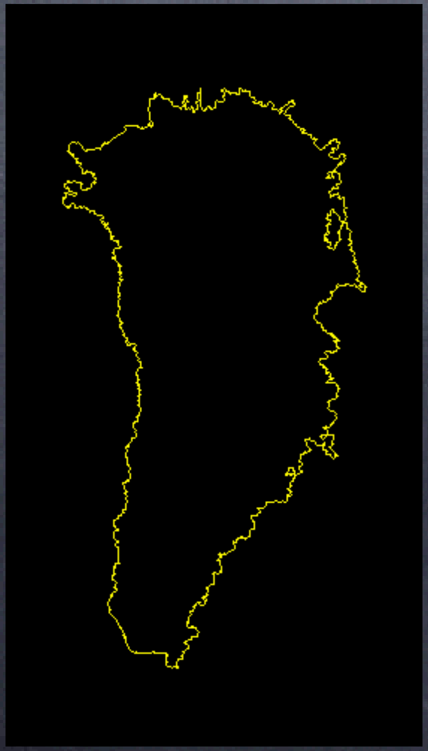
step 1

Determine density function



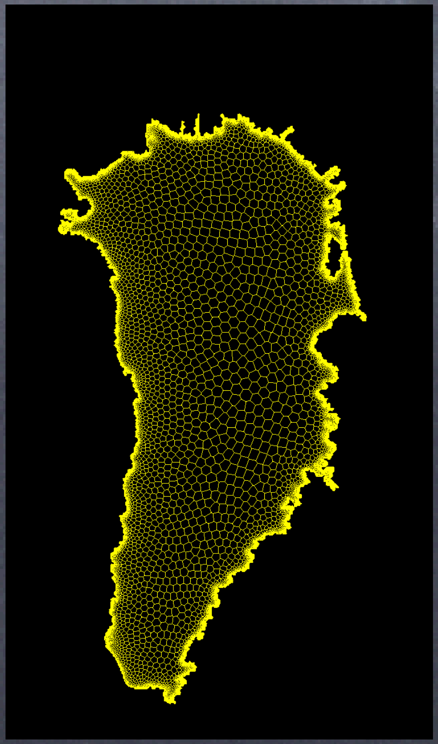
step 2

Determine boundary

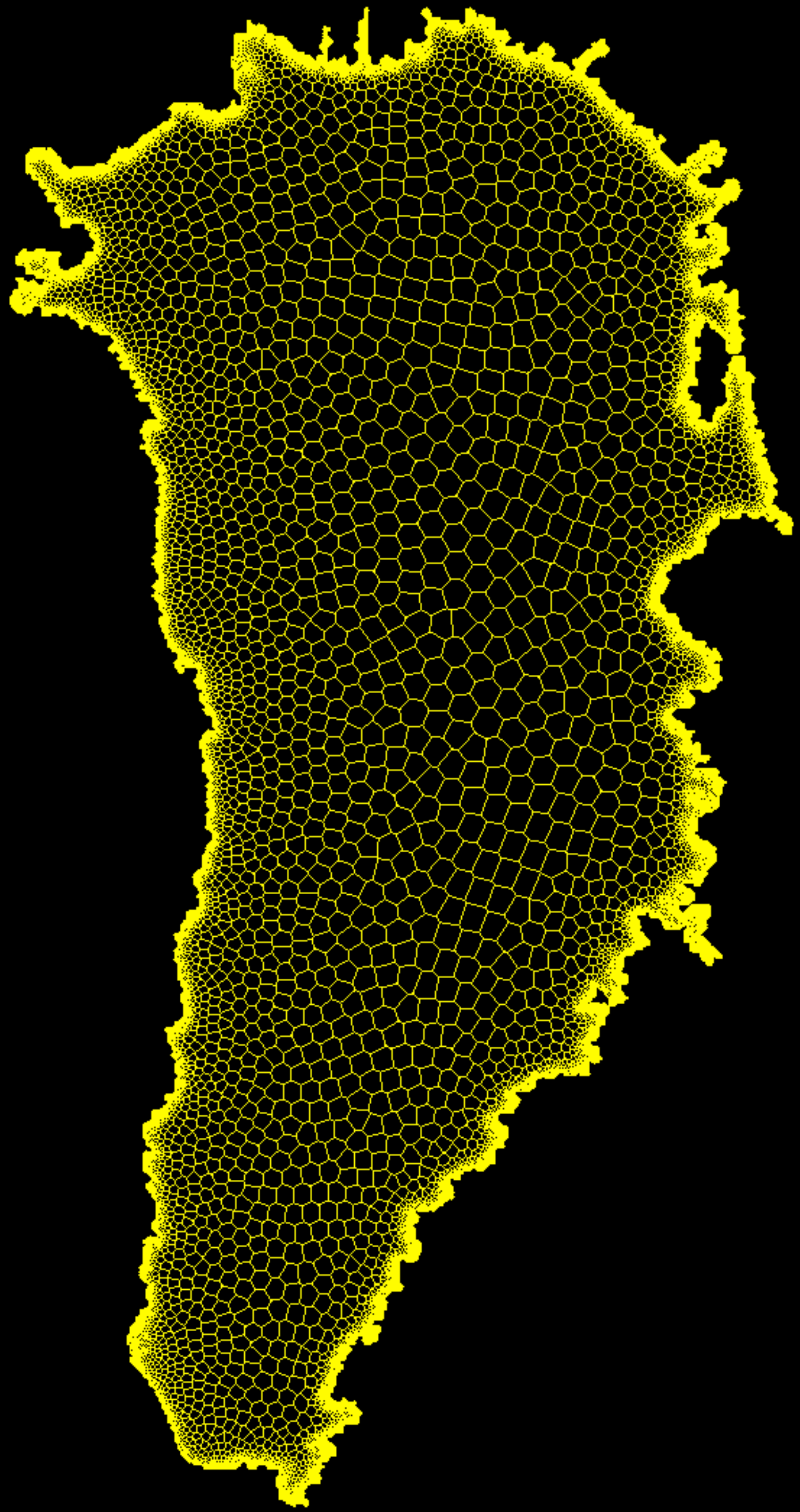


step 3

Build Mesh



step 3

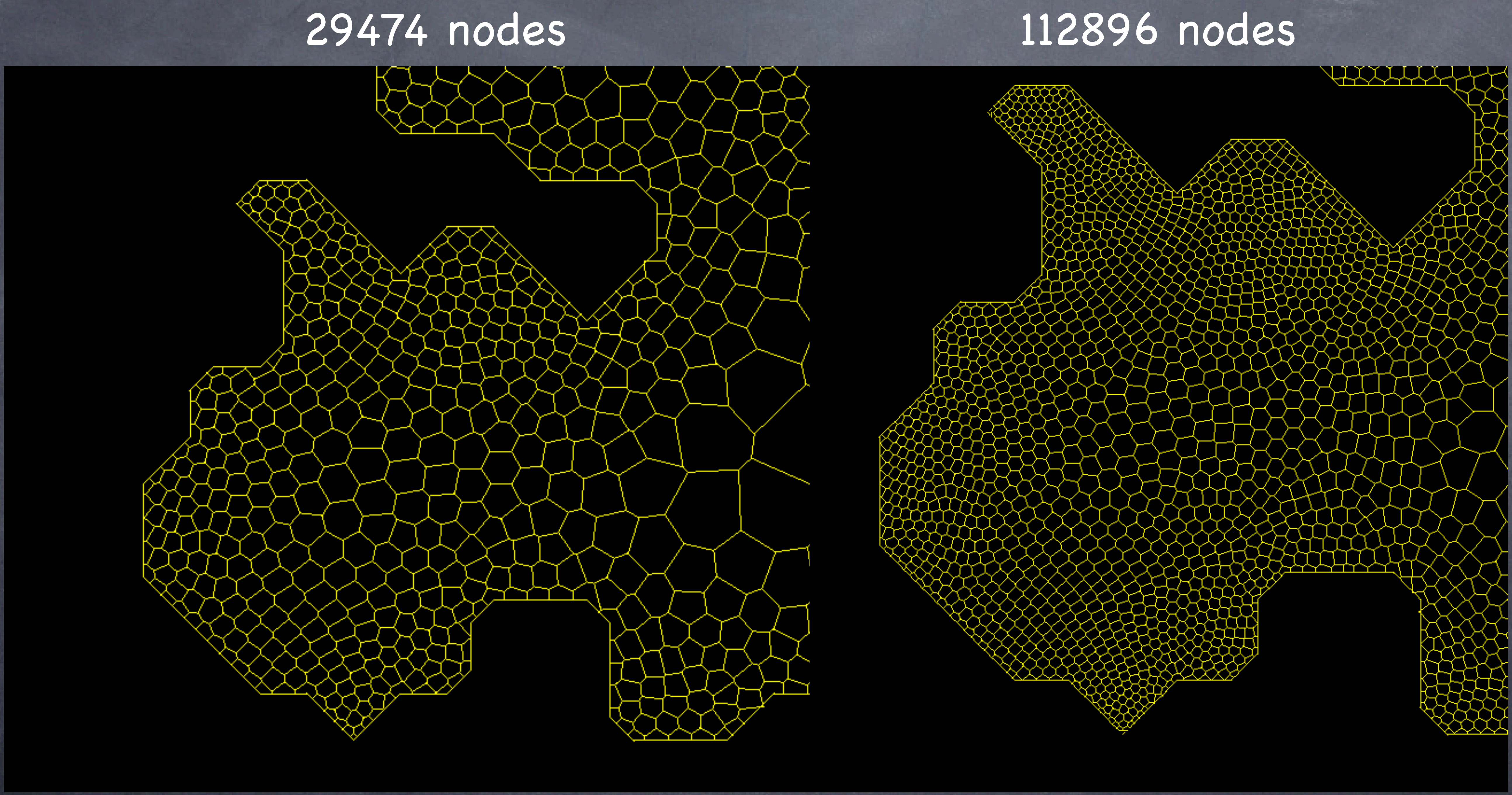


Example Grids: Greenland

Grid Description

# of cells	neighbor distance
43	< 1.0 km
30822	< 2.5 km
70847	< 5.0 km
74609	< 7.5 km
75844	< 10.0 km
80094	< 25.0 km
83422	< 50.0 km
83422	< 75.0 km
83422	< 100 km

And if we add more nodes ...



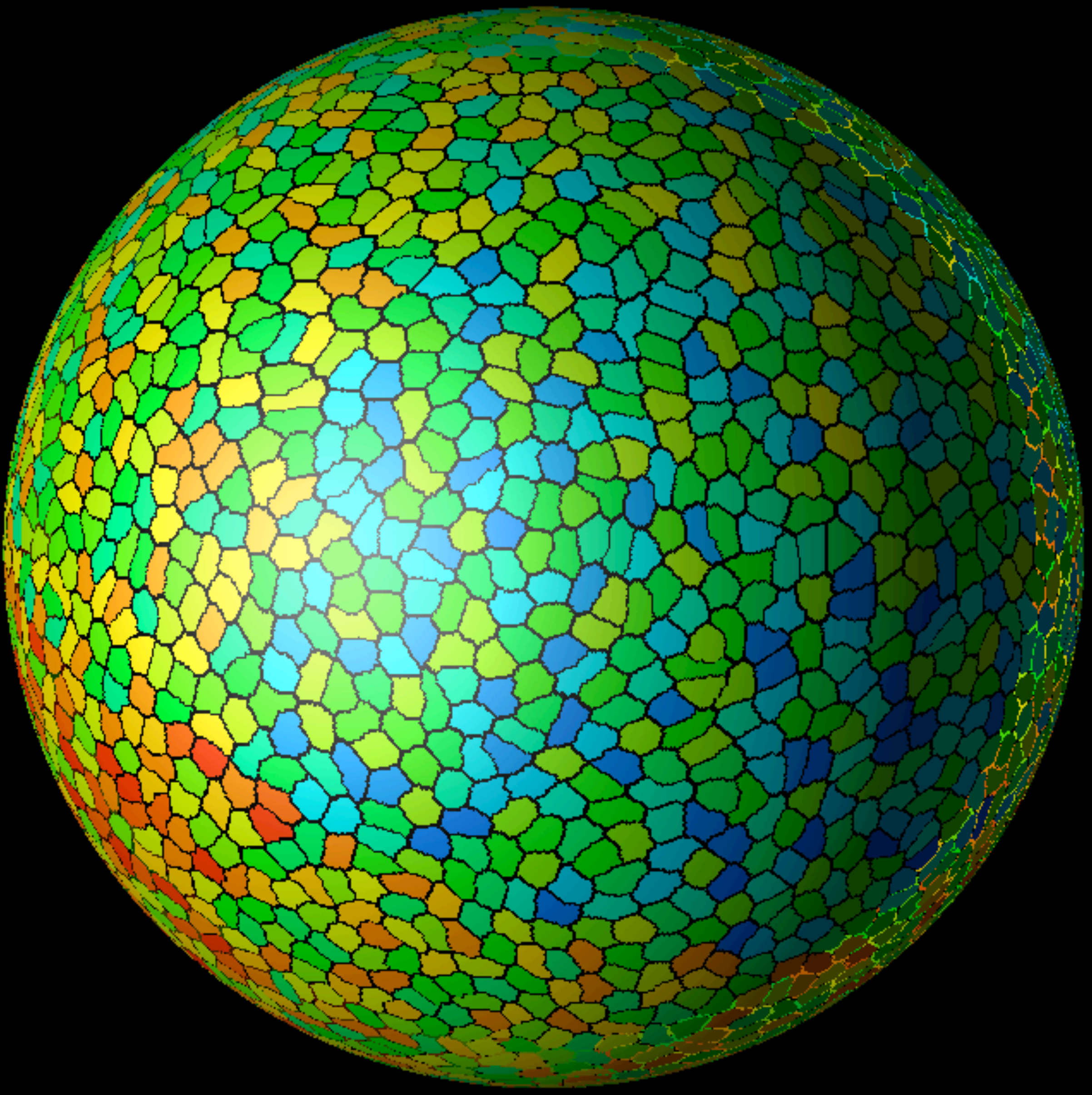
the grid becomes smoother.

A Design for High Performance Computing
From the outset we are targeting efficient performance on 10,000 to 100,000 processors.

Here we taken a global grid of 655362 cells (~30 km resolution) and separated it into 2562 individual blocks.

These blocks are created to balance the work-per-block and to minimize the amount of information that must be communicated between blocks.

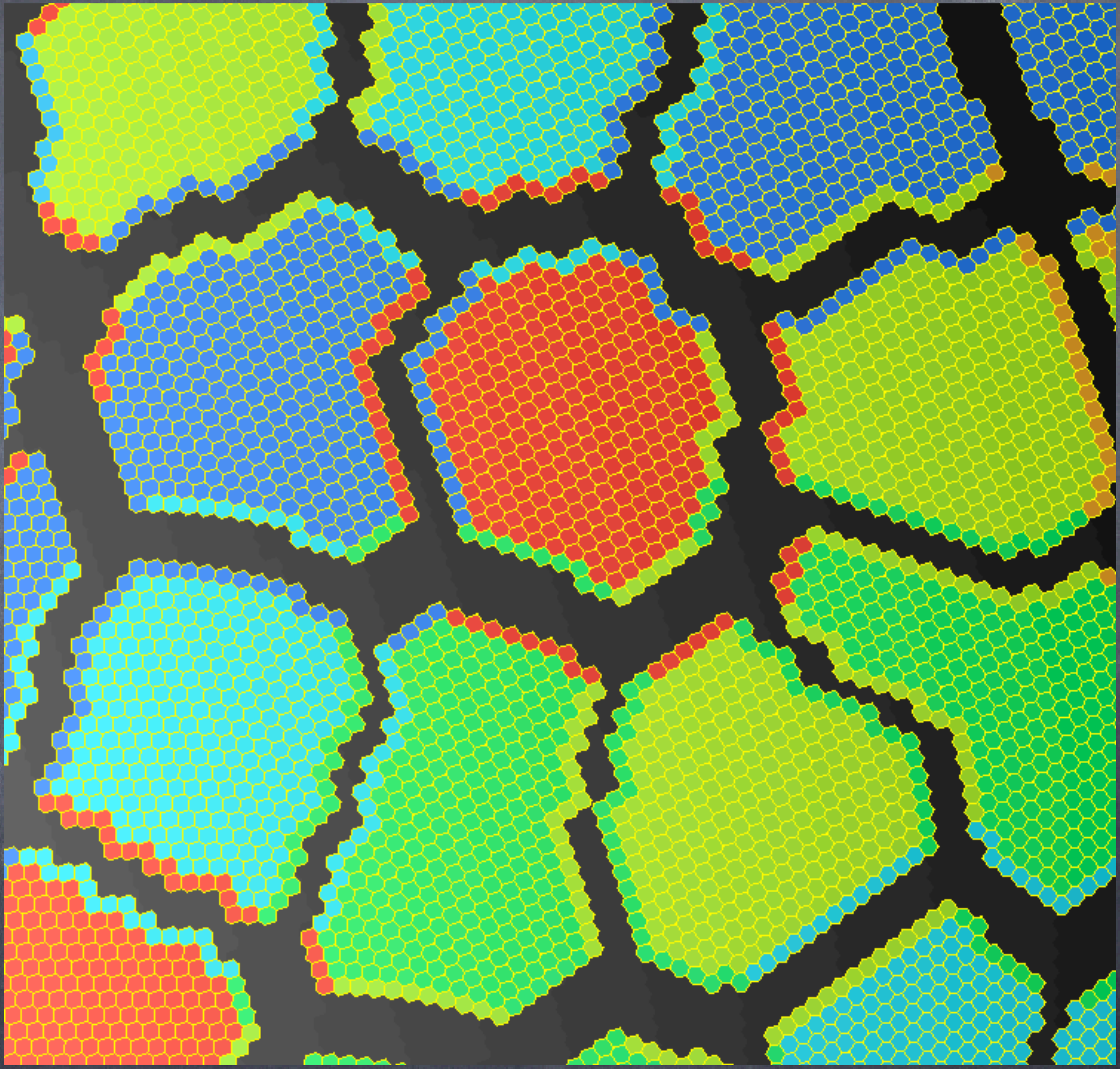
We can assign an arbitrary number of blocks per processor and, thus, support two types of parallelism within this framework (i.e. distributed memory across nodes and shared memory within a node).



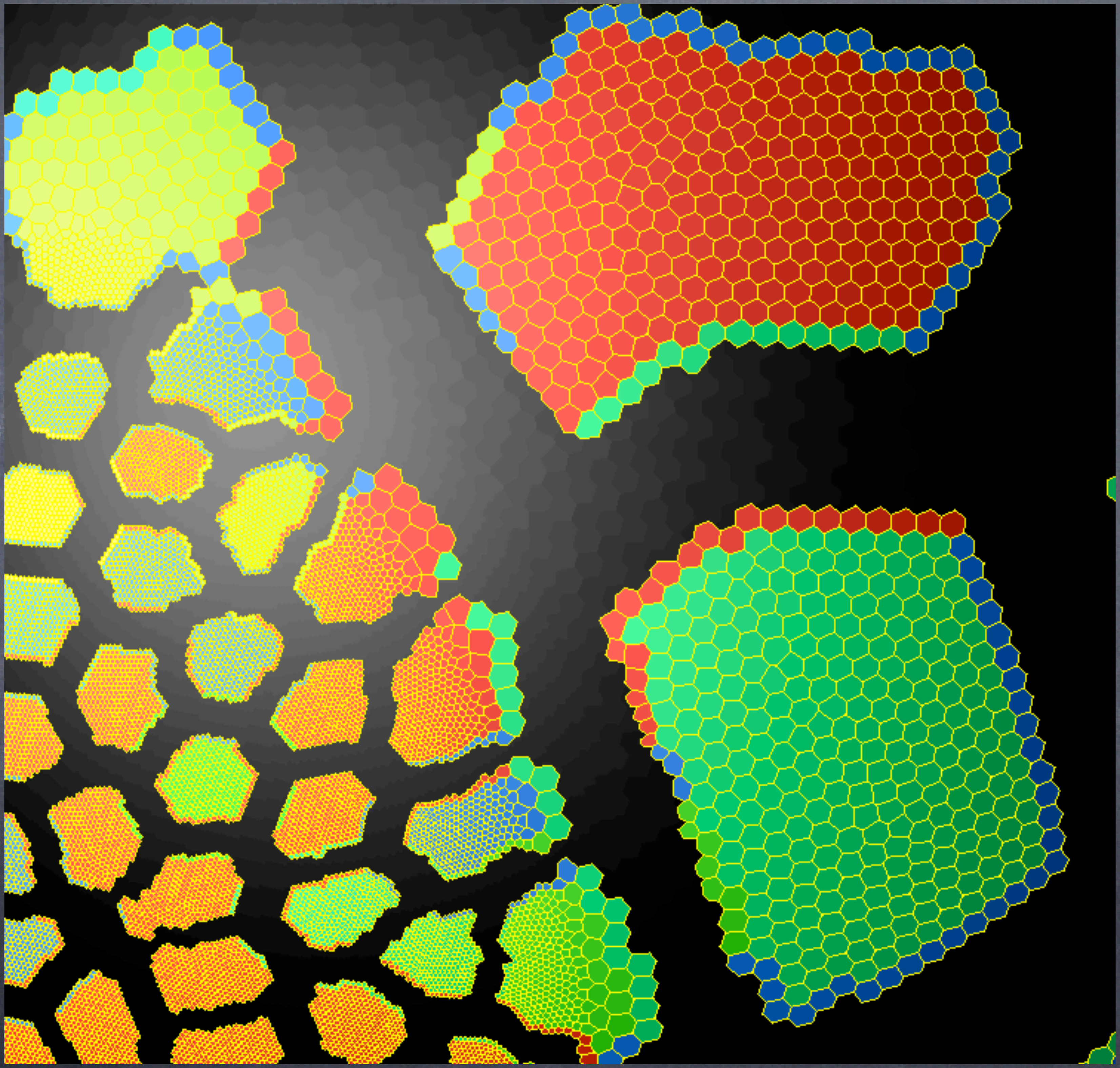
A global 30 km broken into 2562 blocks

A Design for High Performance Computing

Close-up of block decomposition showing "ghost" cell data that indicates interblock communication.



And on a nonuniform SCVT ....

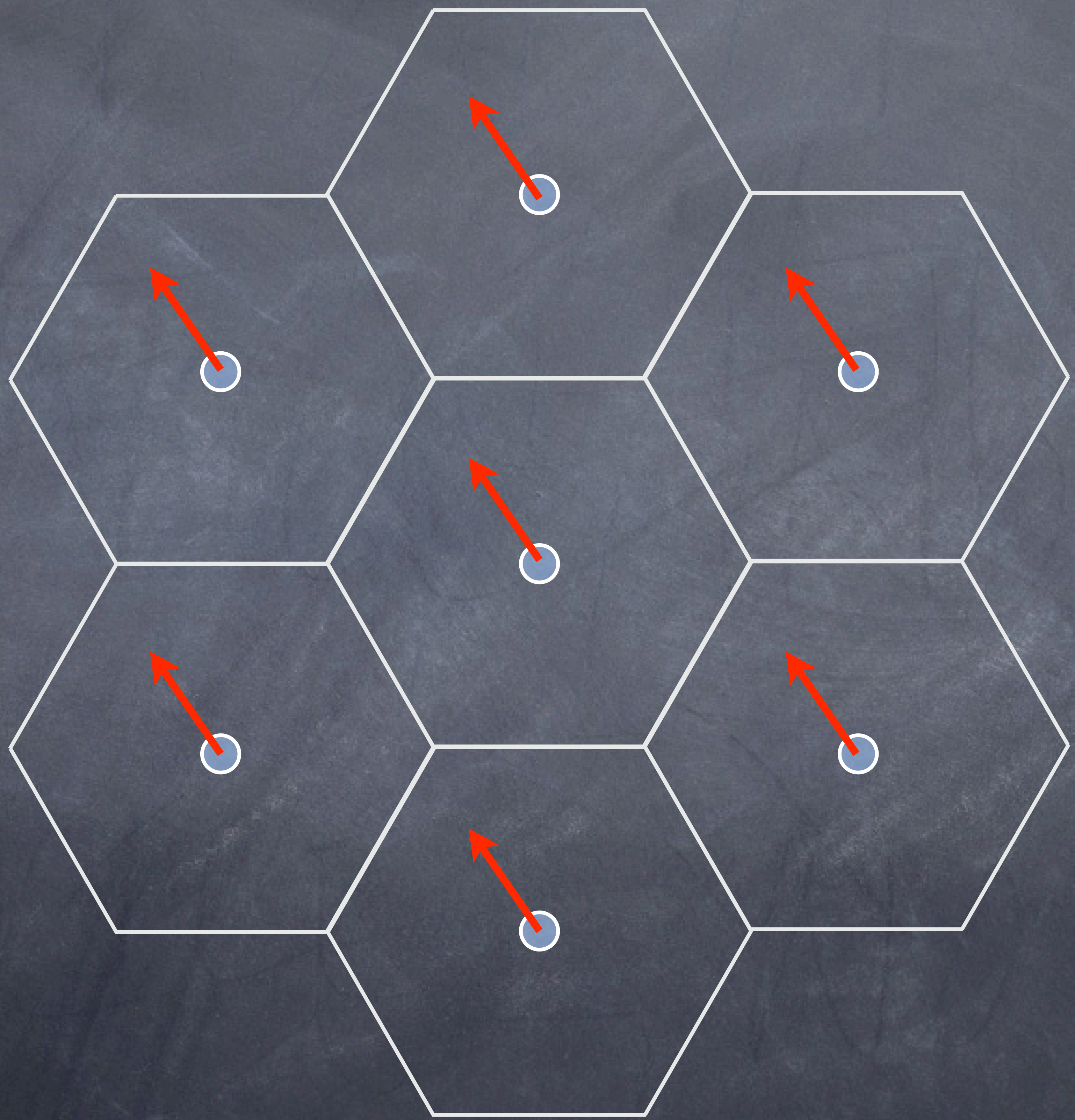


Numerical Methods on variable resolution SCVT grids.

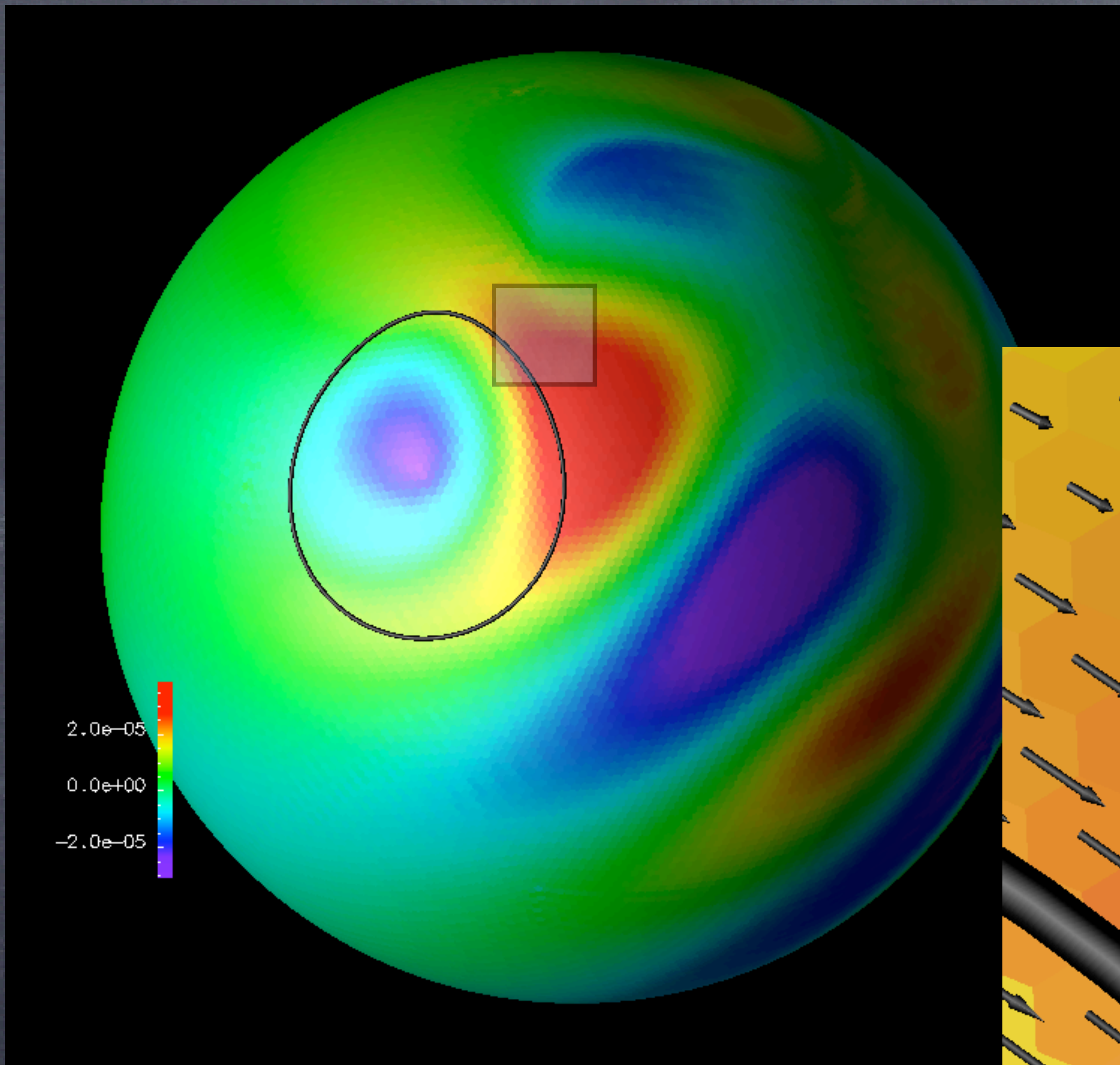
Example for demonstration:
Shallow-water test case #5

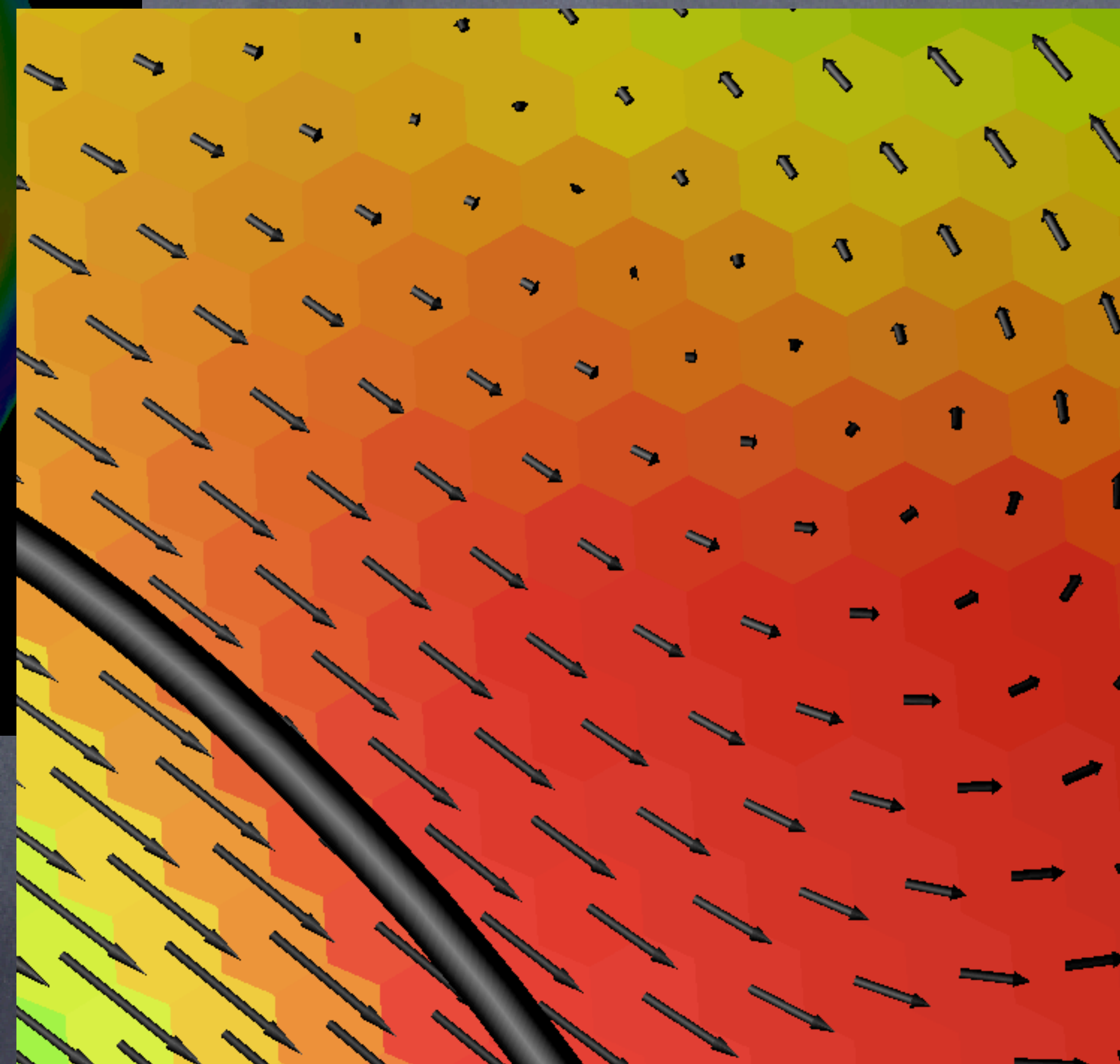
Geostrophically-balanced flow confronts a 2 km mountain at t=0.

Method #1: Collocated Hexagonal Grid

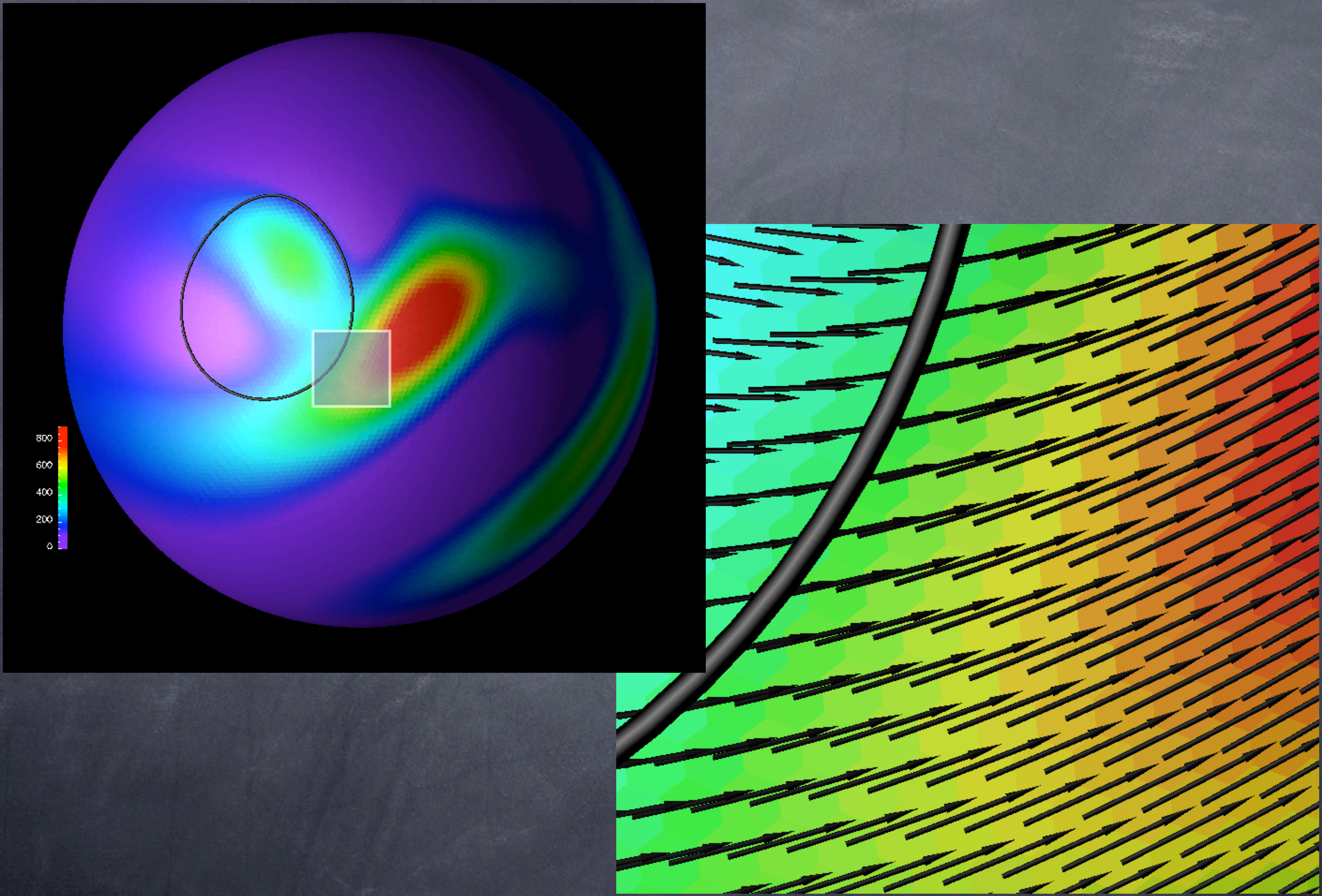


Relative Vorticity day 10 (collocated A-grid, uniform mesh)

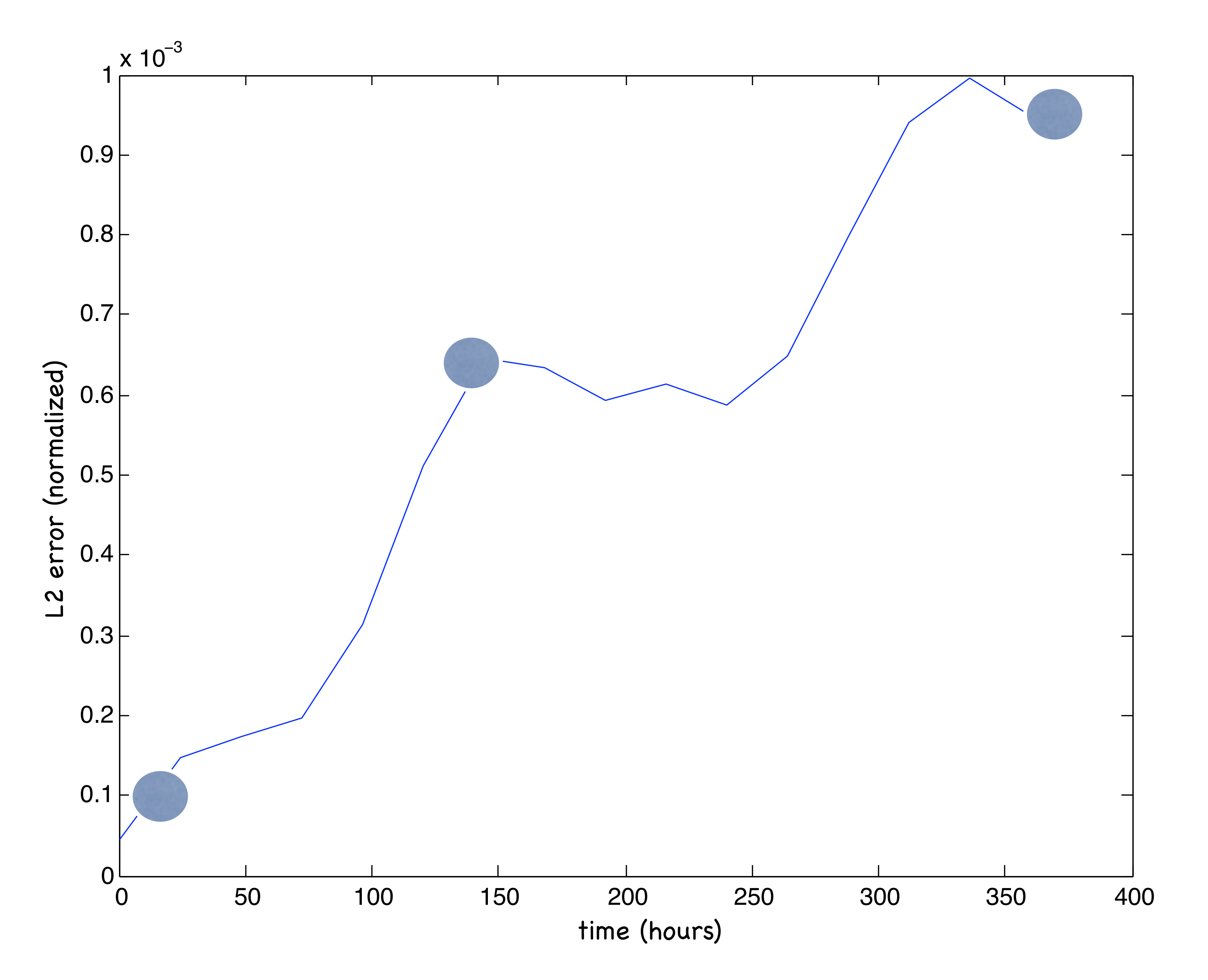


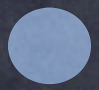


Kinetic Energy day 10 (collocated A-grid, uniform mesh)



L2 norm, SWTC#5 (40962, glevel 6)



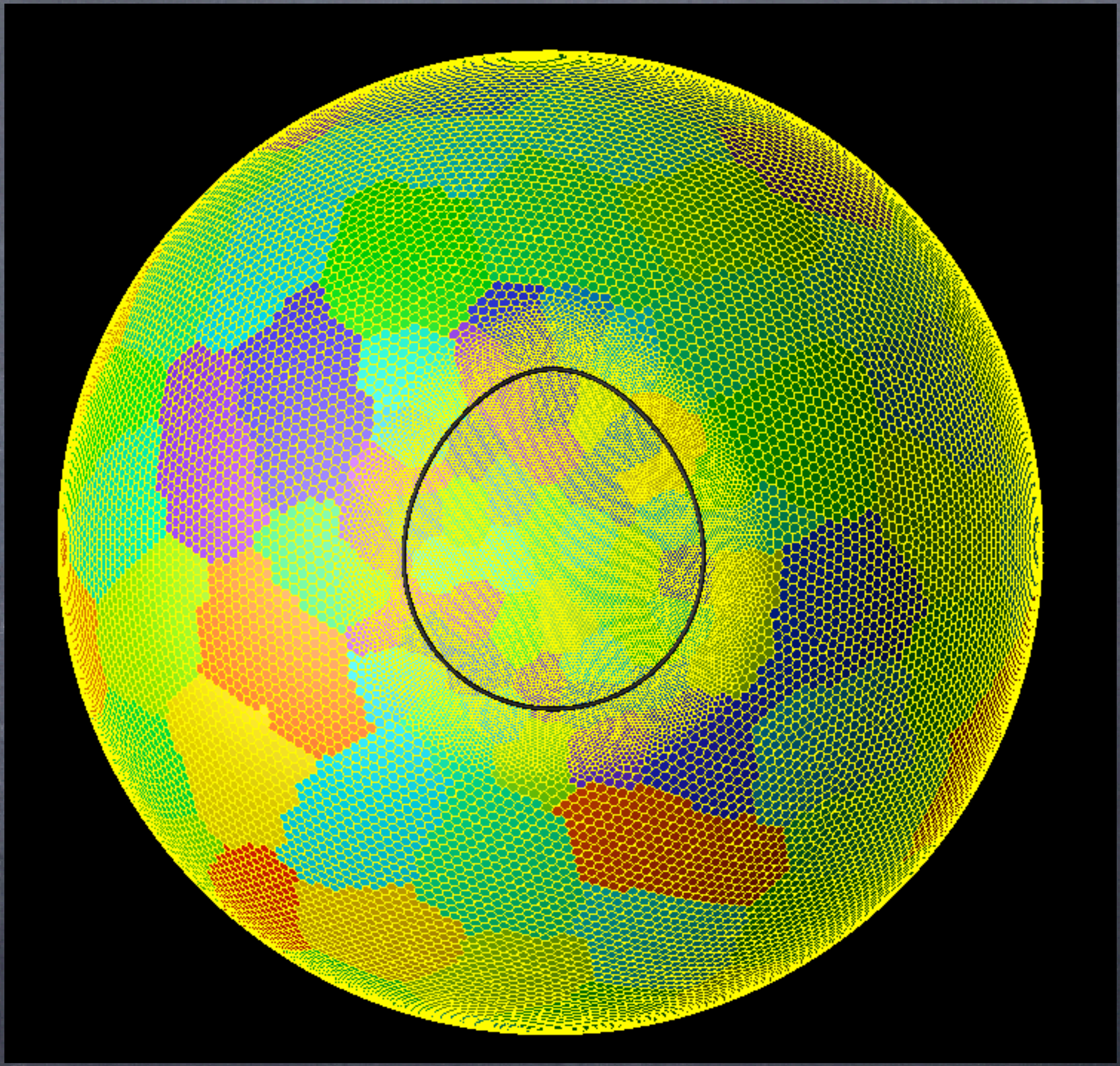


Data from the literature

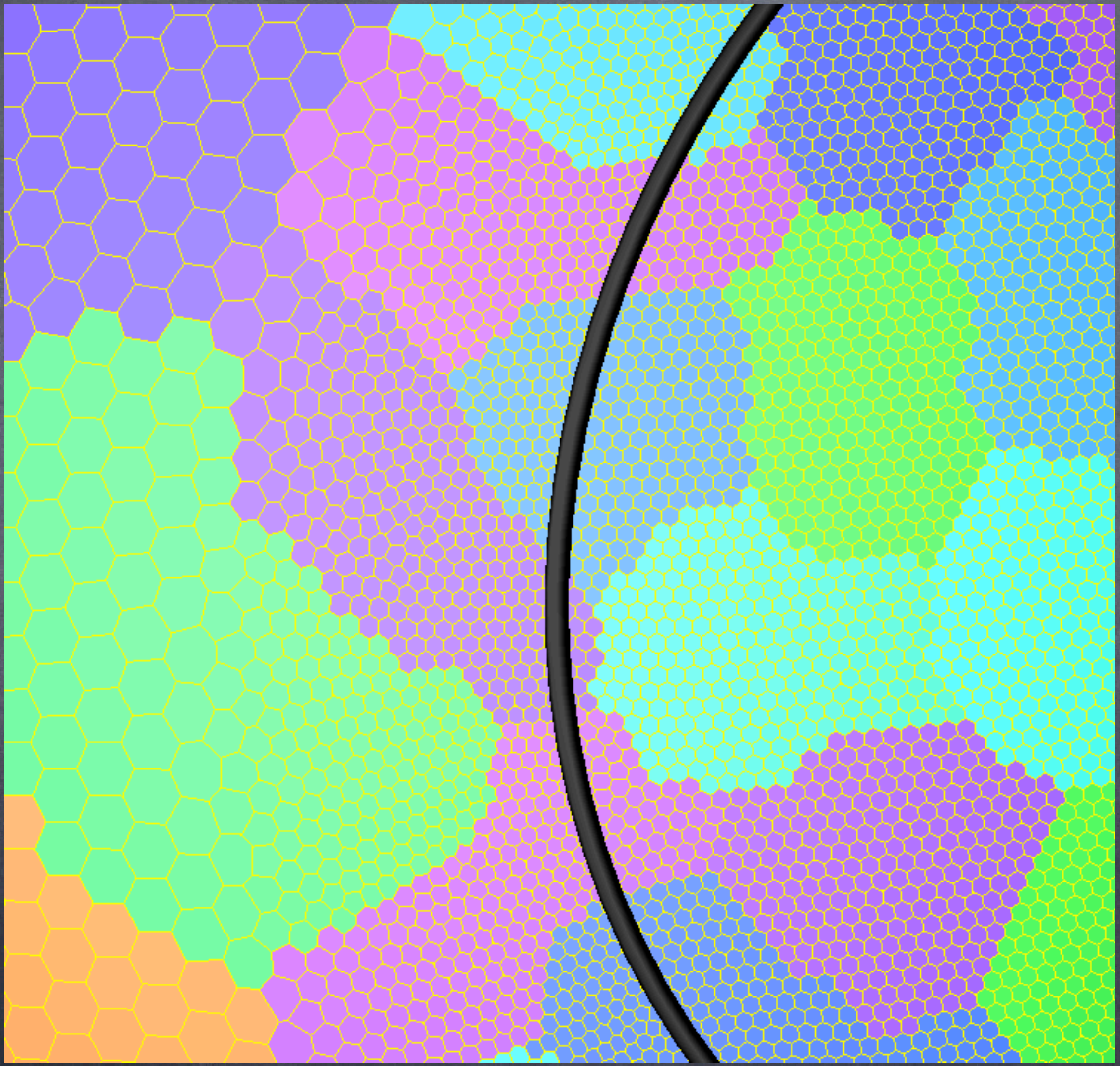
Same number of cell (40962), but focused in the vicinity of the topographic forcing.

Apply the same method to the same problem, but focus the resolution (and, thus, computational resources in the vicinity of the forcing.

Grid spacing in "high-res" region is about 40 km (a factor of three higher than elsewhere).



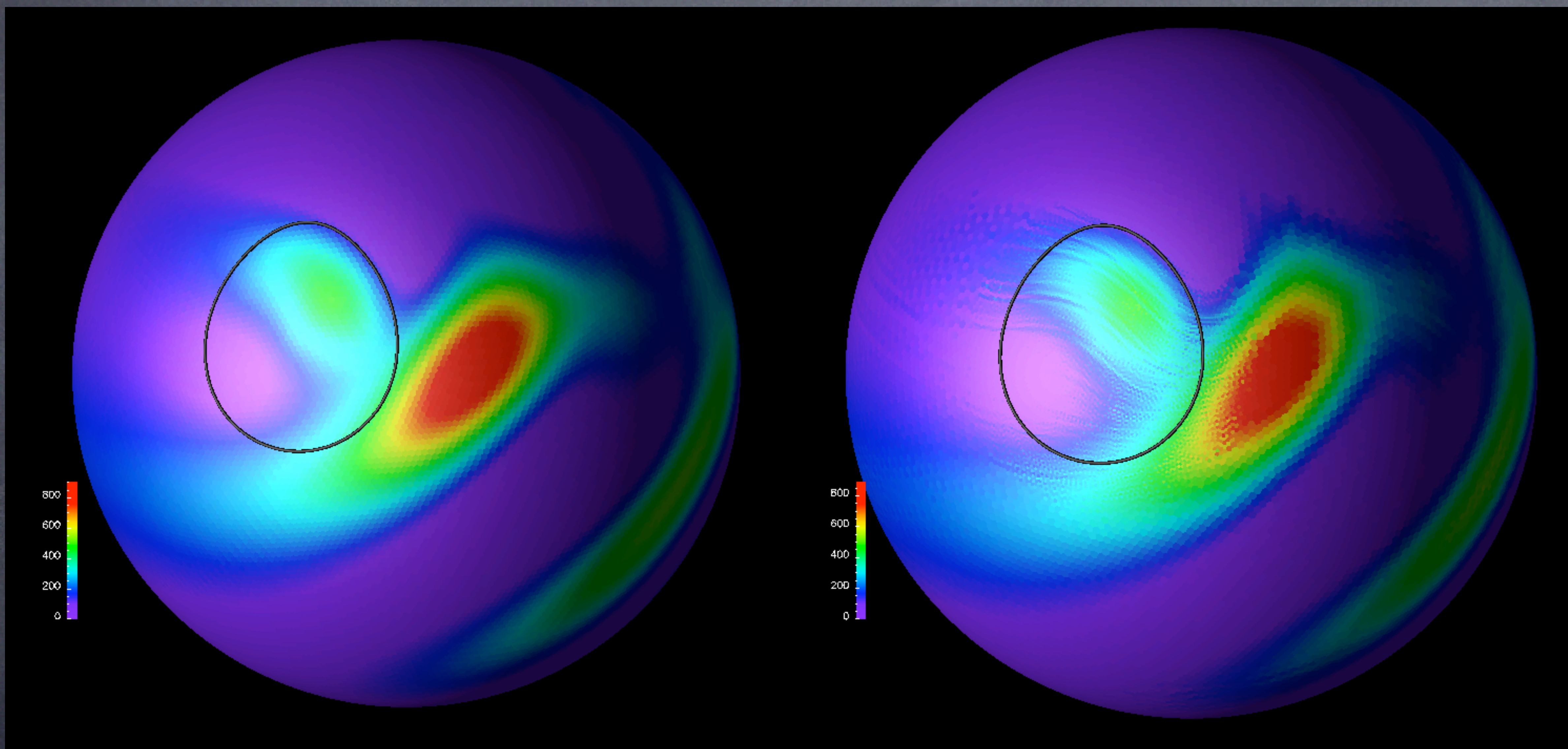
A closer look at the transition region:
Grid spacing differs by approximately a factor of three.



Kinetic Energy

A-Grid, Uniform

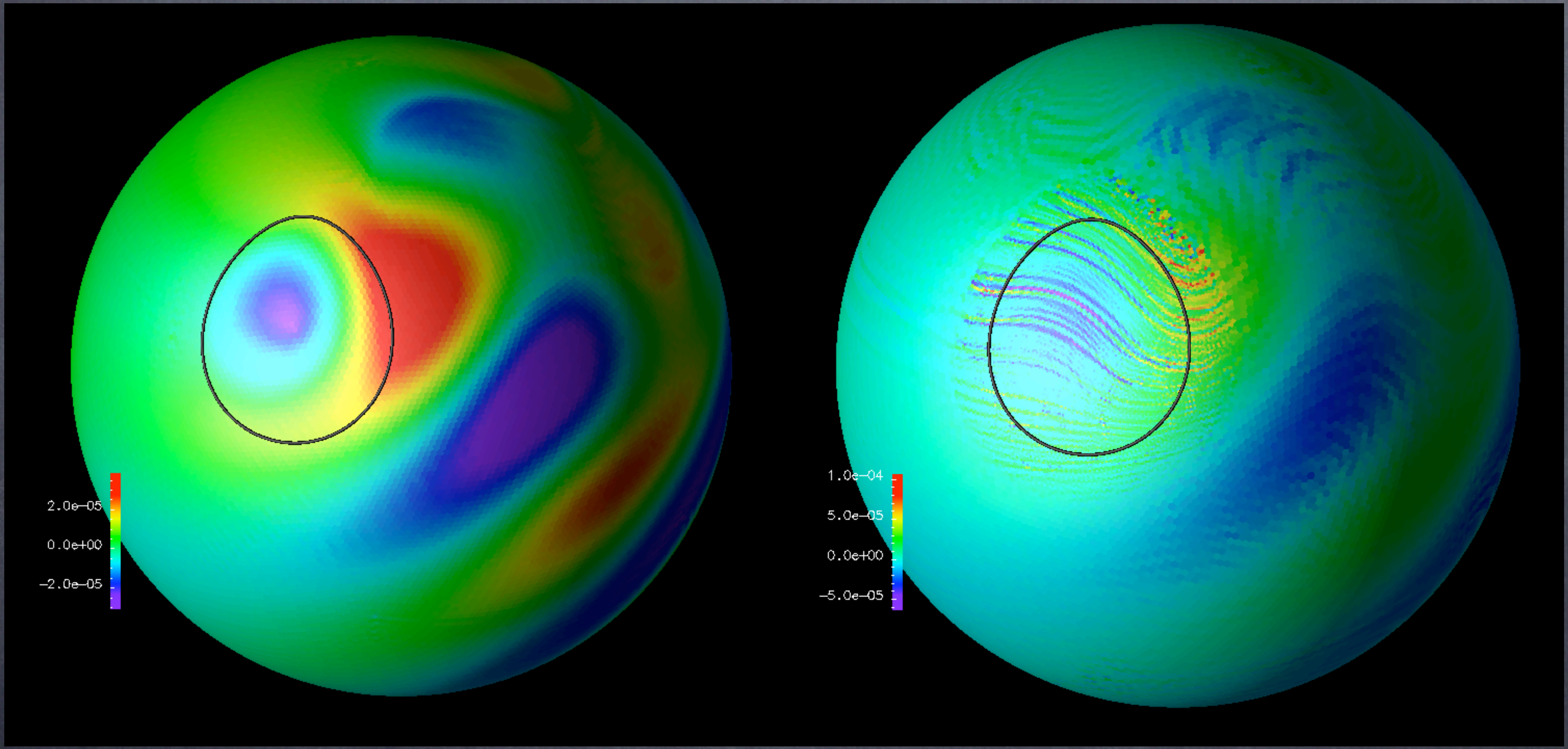
A-Grid, Nonuniform



Relative Vorticity

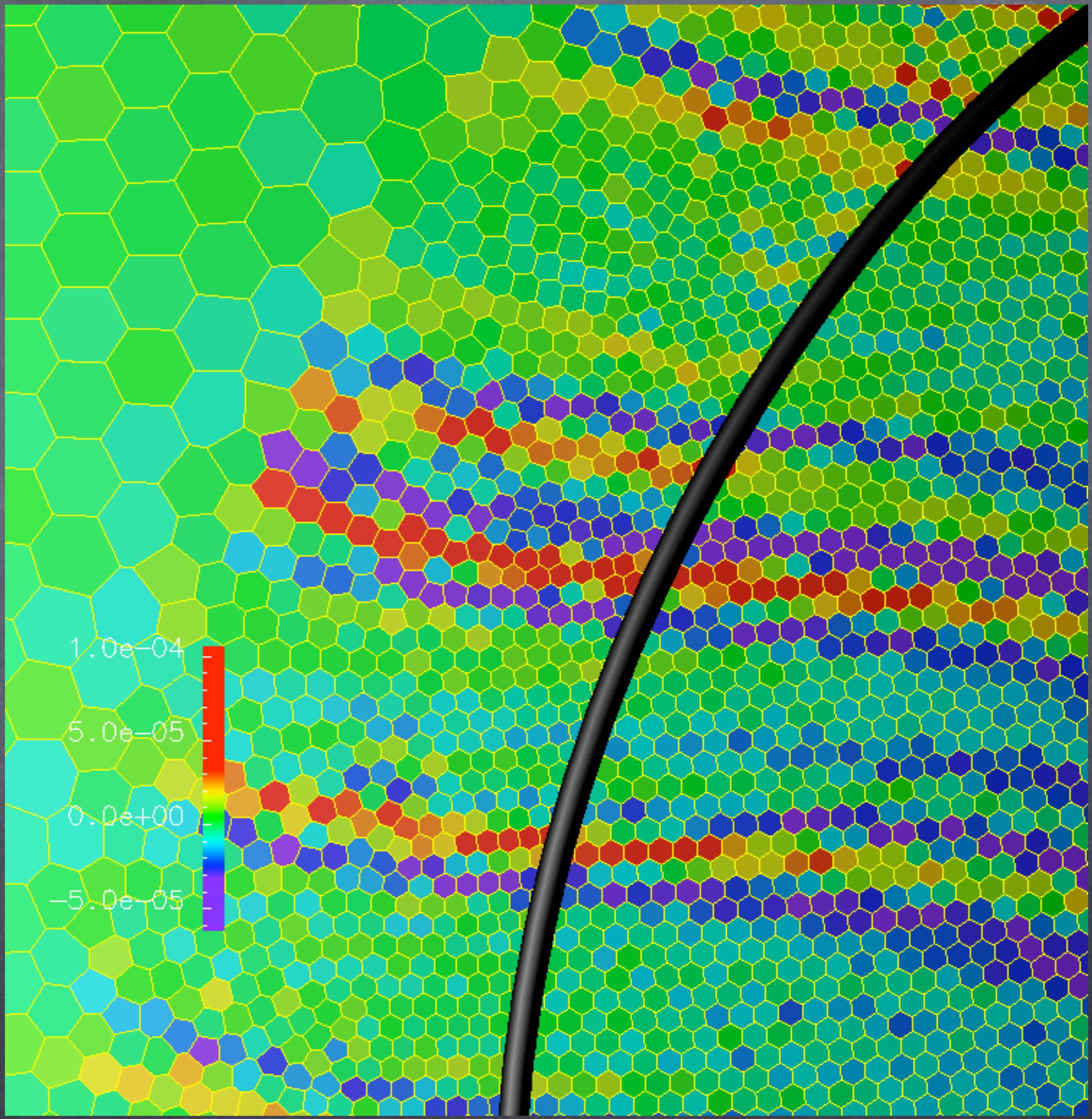
A-Grid, Uniform

A-Grid, Nonuniform

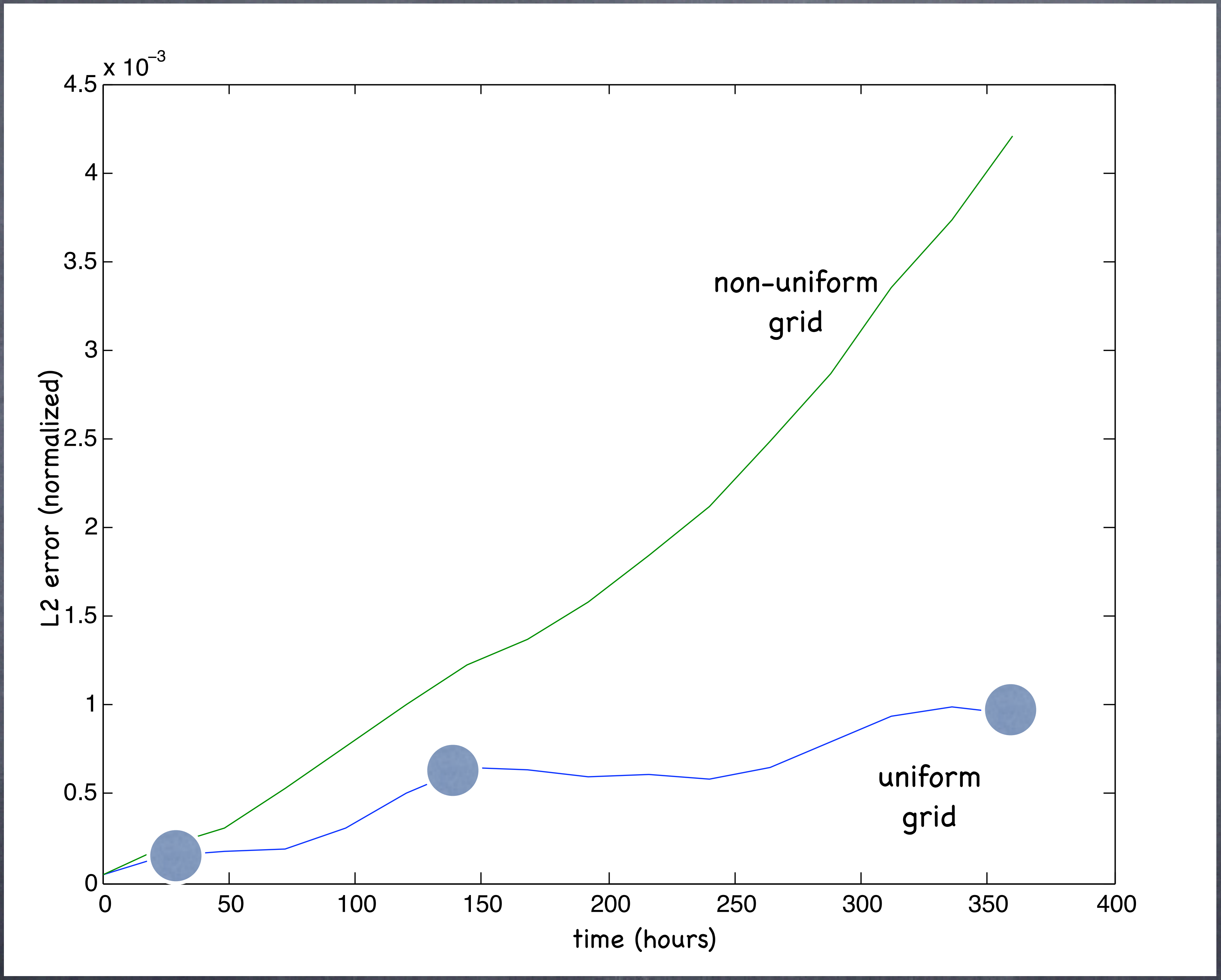


Relative Vorticity

Truncation errors combined with [curl(grad)].ne.0 leads to large vorticity errors in the transition zone that are advected downstream.

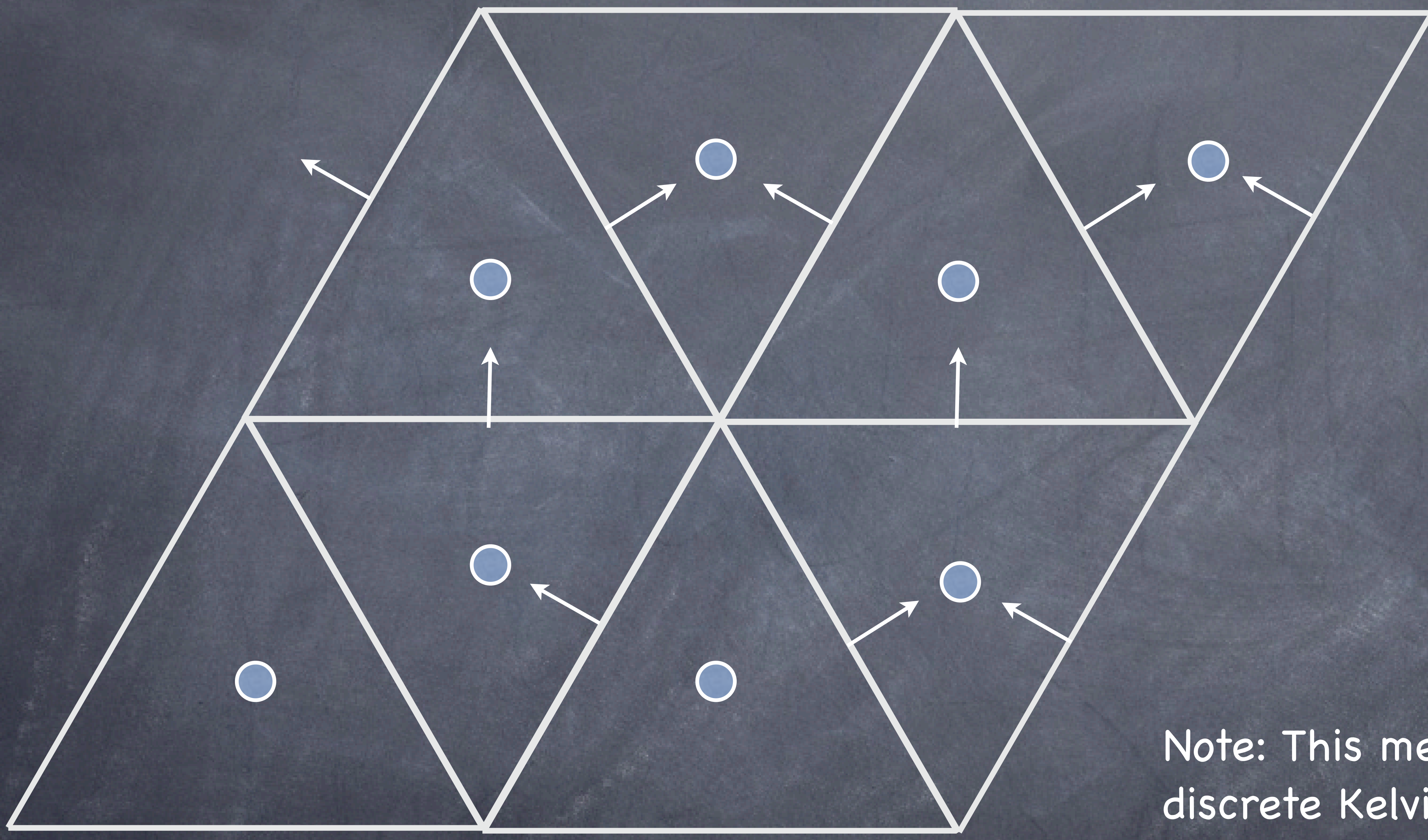


Solution error as a function of time.



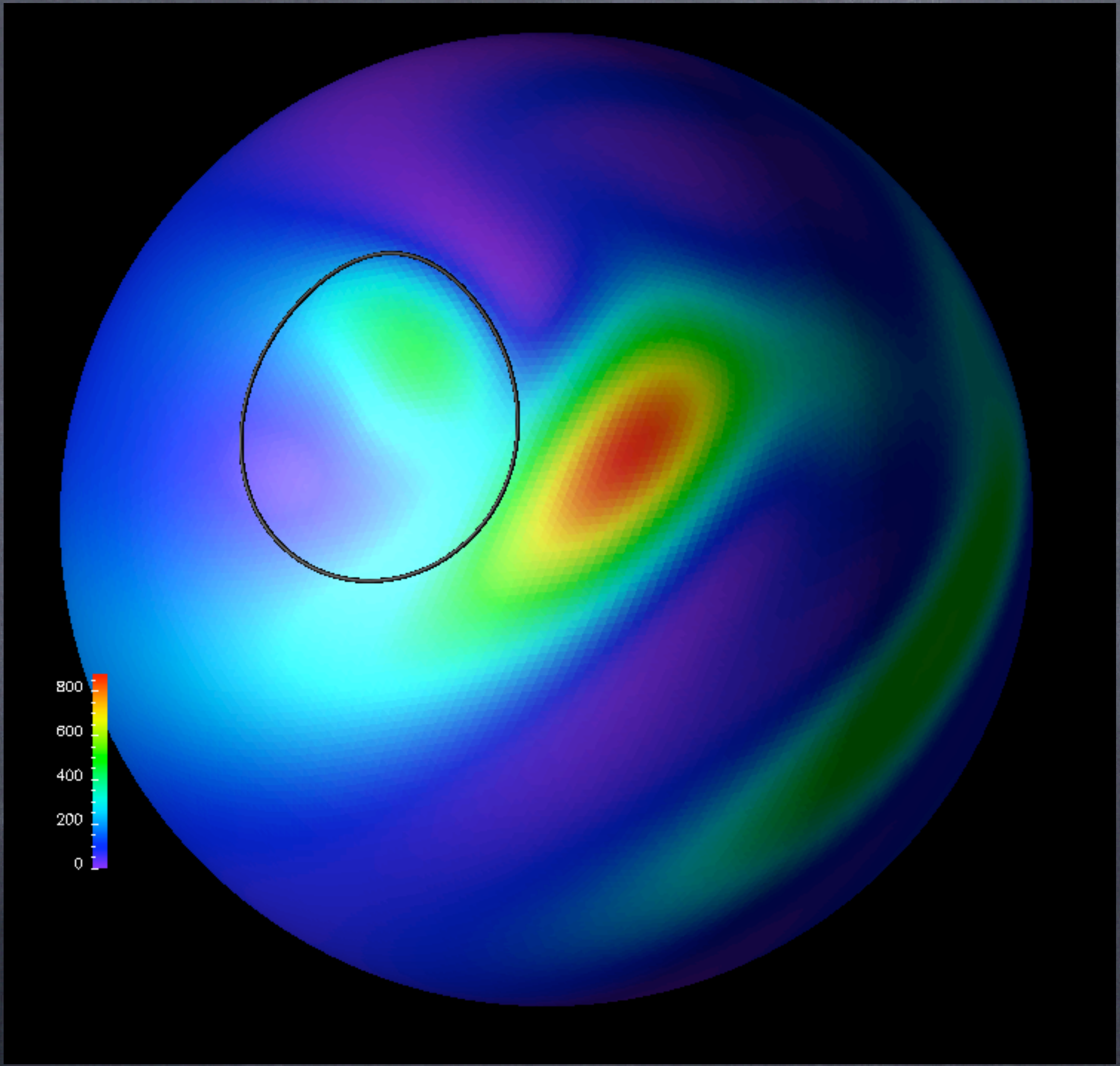
The take-home message here is that methods that produce satisfactory results on a quasi-uniform grid might not be acceptable on a variable resolution grid.

Method #2: C-Grid on Triangles



Note: This method has a discrete Kelvin's Circulation Theorem as discussed on Monday.

Kinetic Energy day 10 (triangle C-grid, uniform mesh)



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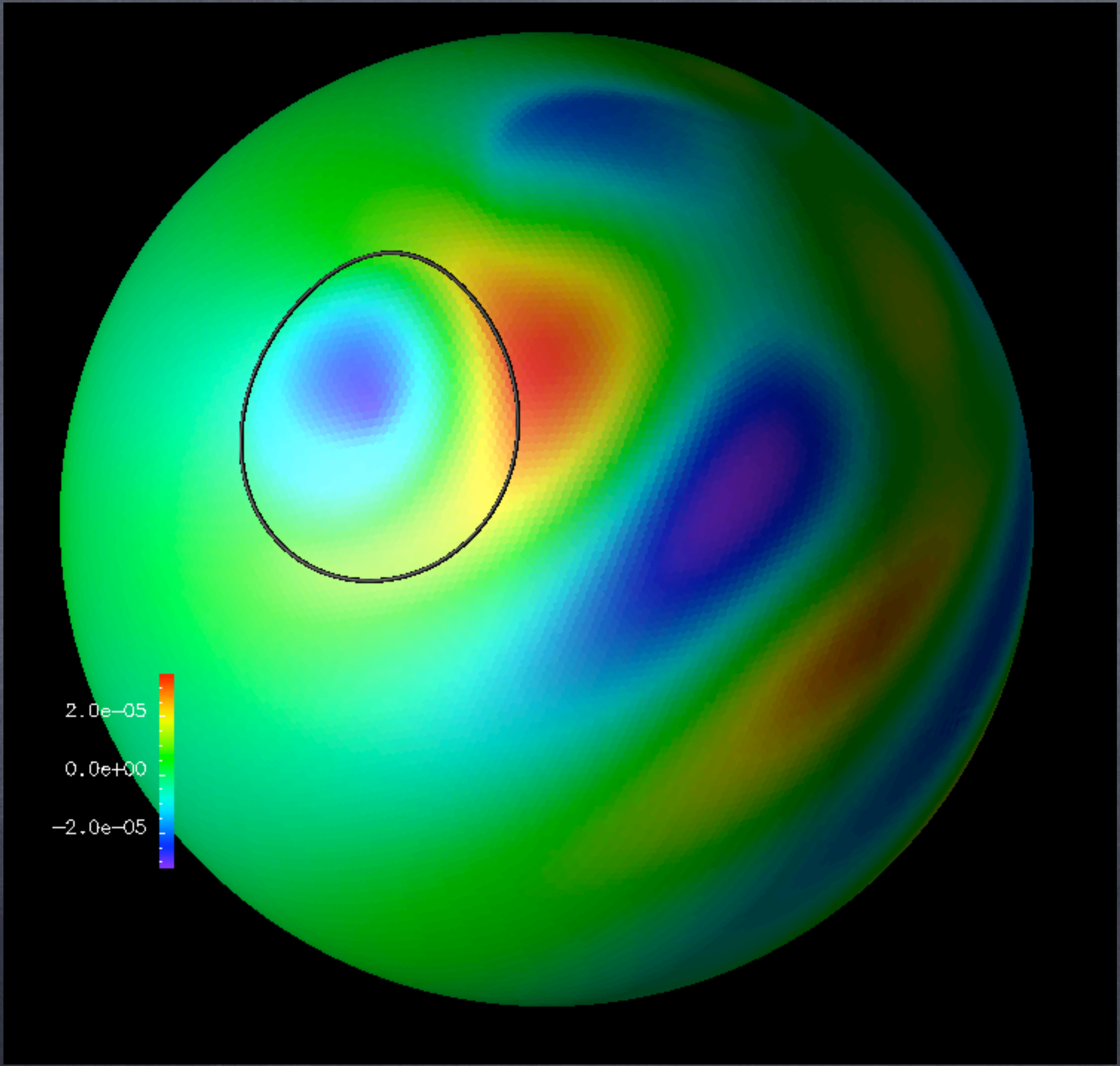

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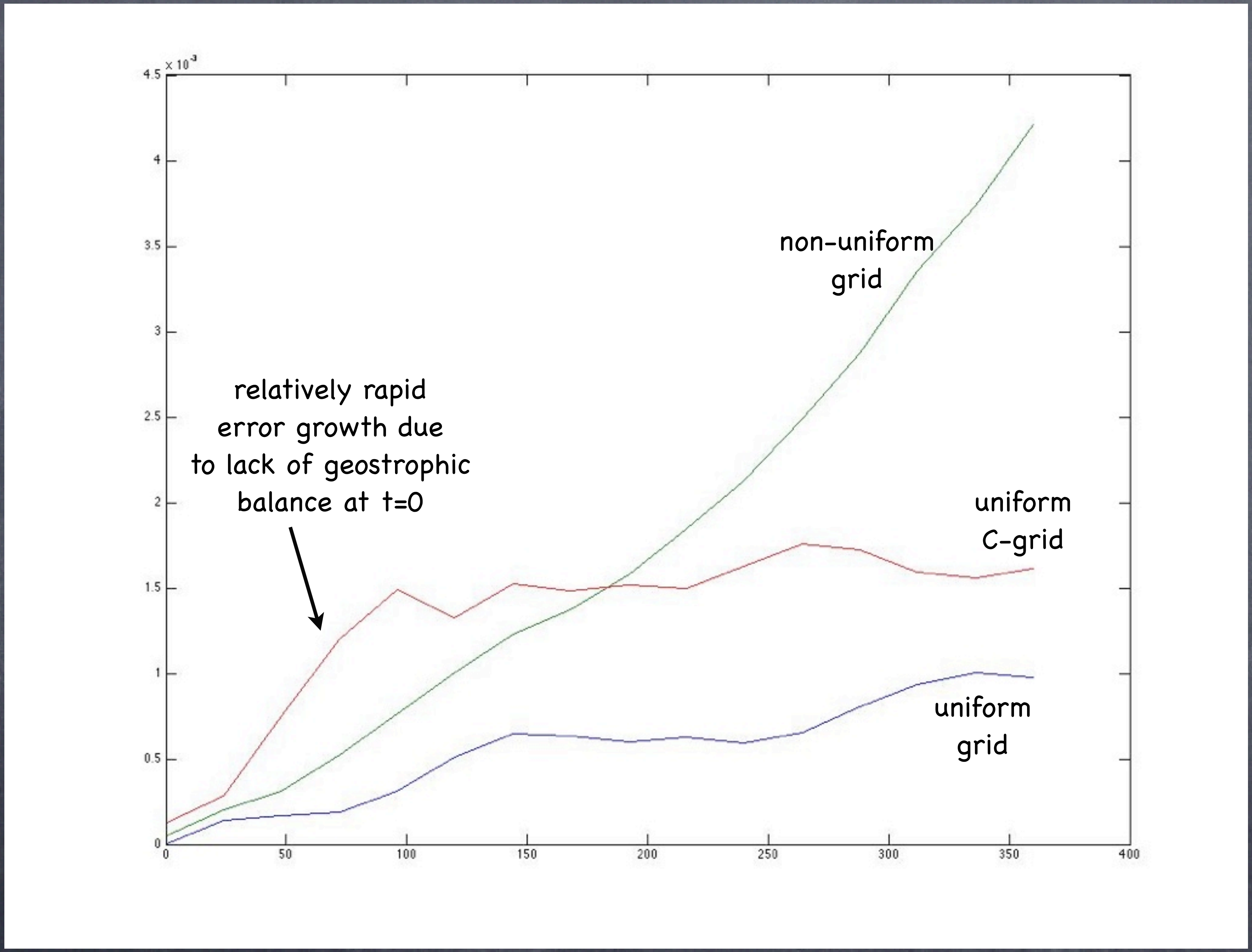

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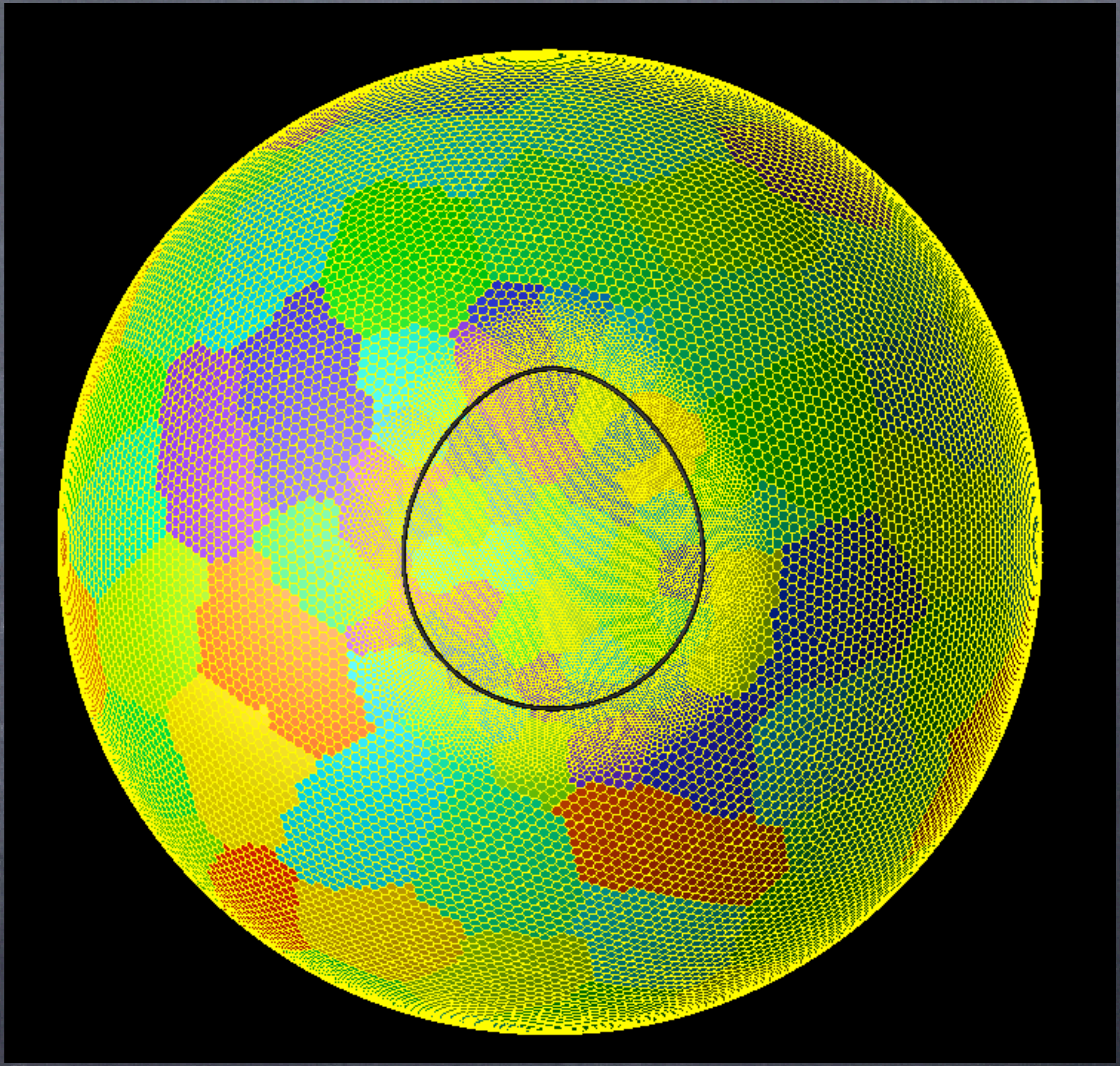

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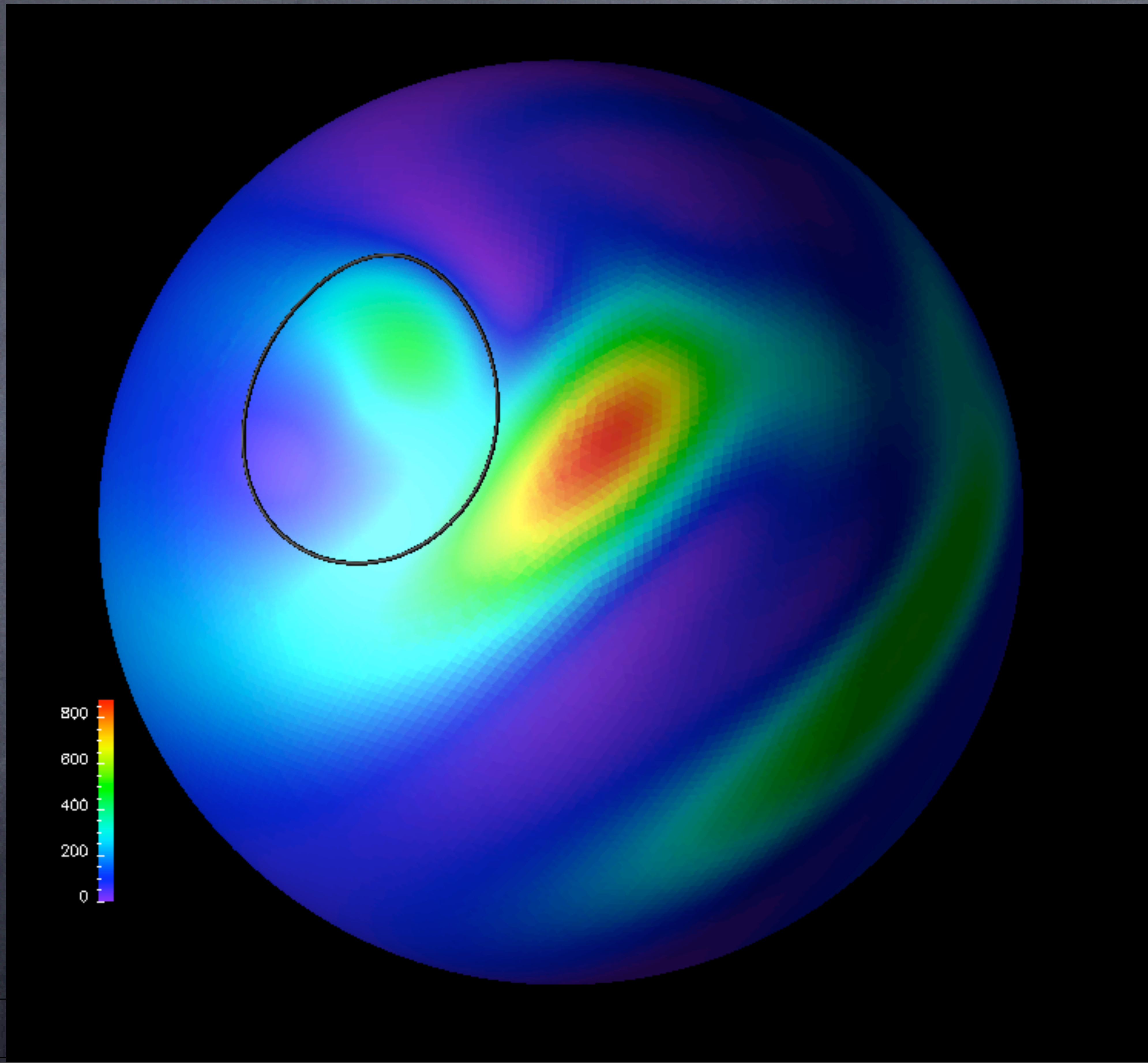

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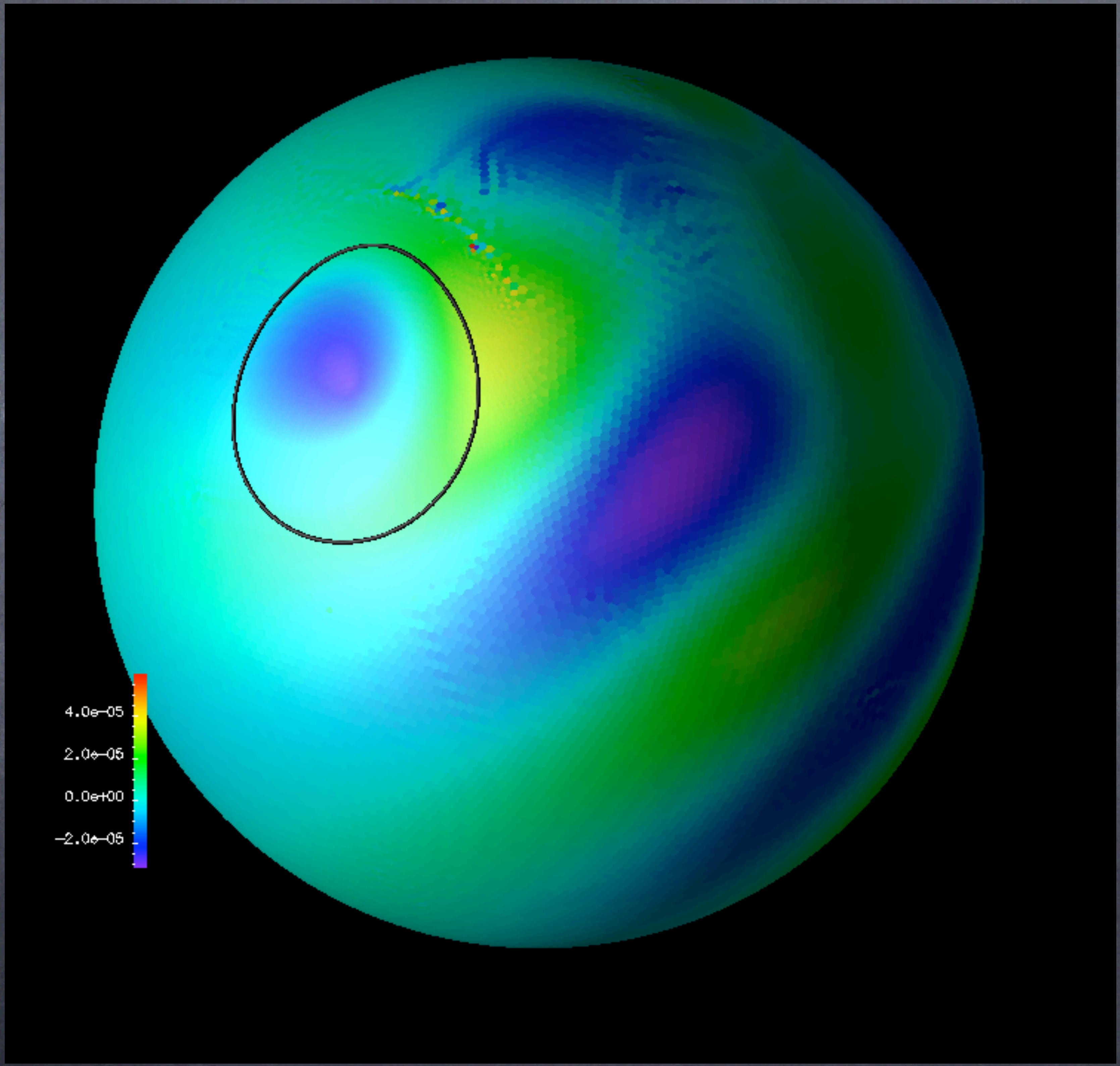
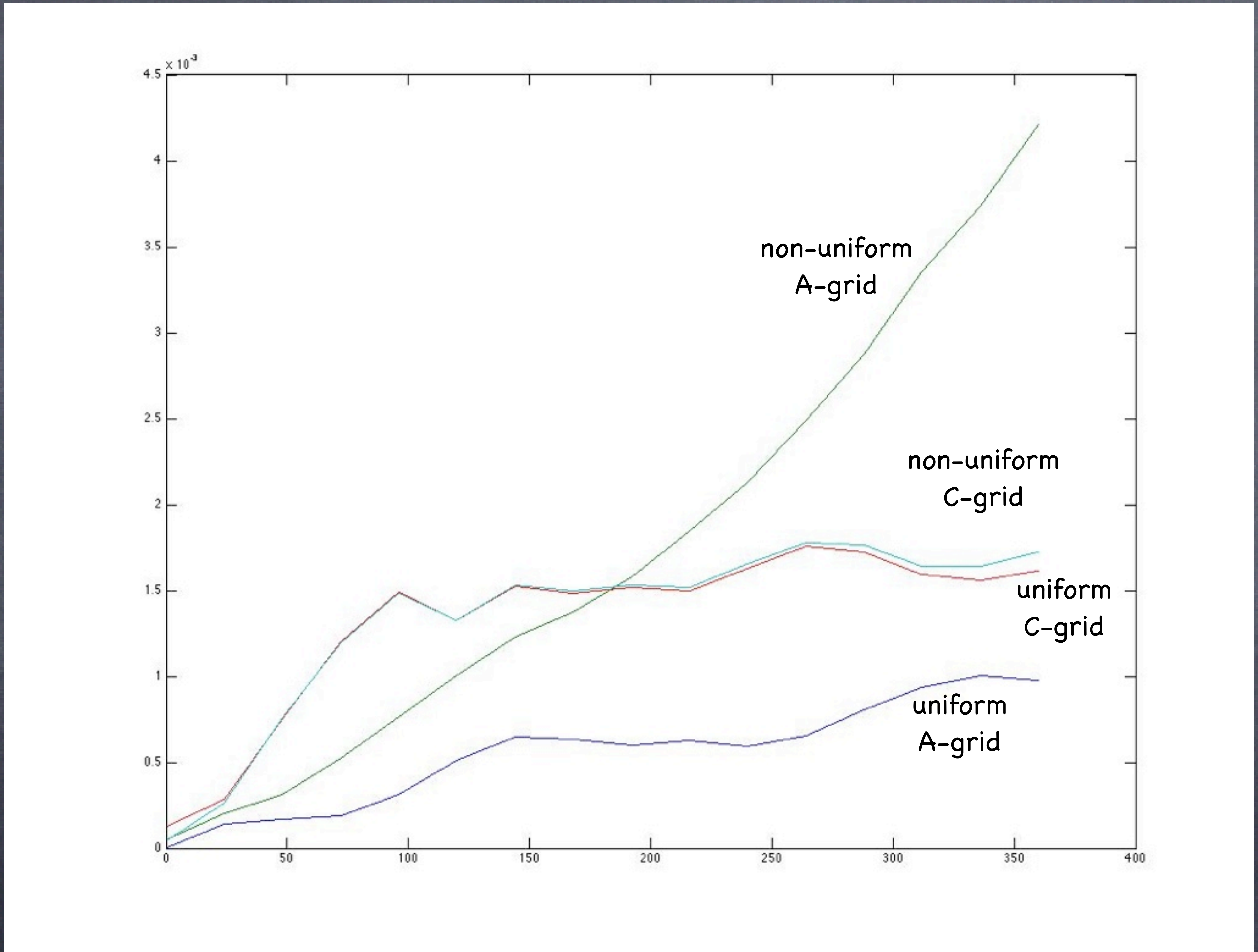
Kinetic Energy day 10 (triangle C-grid, uniform mesh)



Solution error as a function of time.



Summary

Voronoi Diagrams and, in particular, Centroidal Voronoi Diagrams offer a robust approach to tiling the surface of the sphere.

The Delaunay Triangulation is the dual of the Voronoi Diagram, so whether "hexagons" or triangles are your interest, this approach will work.

Centroidal Voronoi Diagrams are particularly well-suited for the generation of smoothing varying meshes, thus providing a possible alternative to traditional nesting approaches.

Numerical techniques that work robustly on variable resolution meshes need attention.