Basic Dynamics - I

Monday 2 June, 2008

Bibliography

- An Introduction to Dynamic Meteorology, J.R. Holton, Elsevier / Academic Press
- Atmosphere Ocean Dynamics, A.E. Gill, Academic Press
- Geophysical Fluid Dynamics, J. Pedlosky, Springer
- Lectures on Geophysical Fluid Dynamics, R. Salmon, OUP

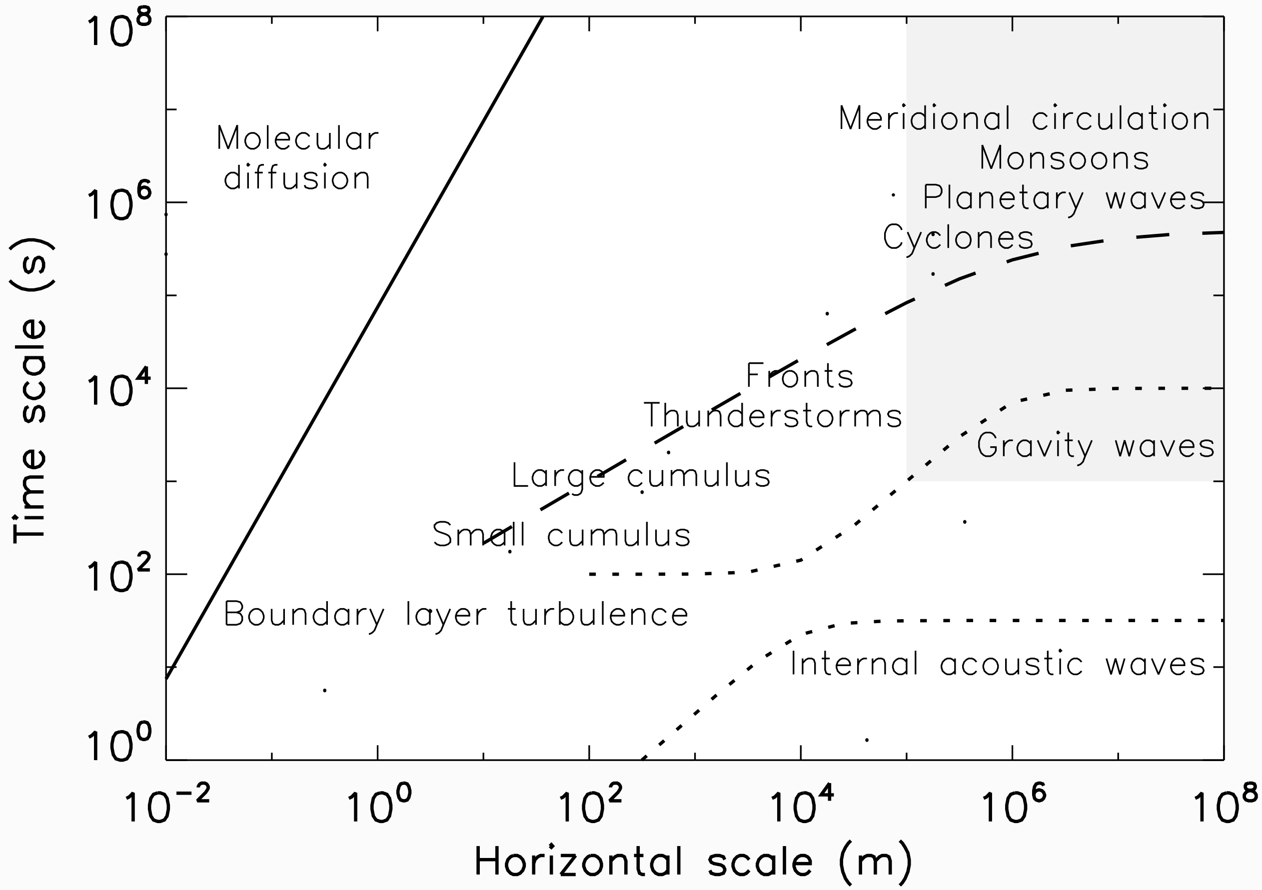
Outline

- The multiscale nature of atmospheric dynamics
- Dynamics in a rotating frame
- Governing equations

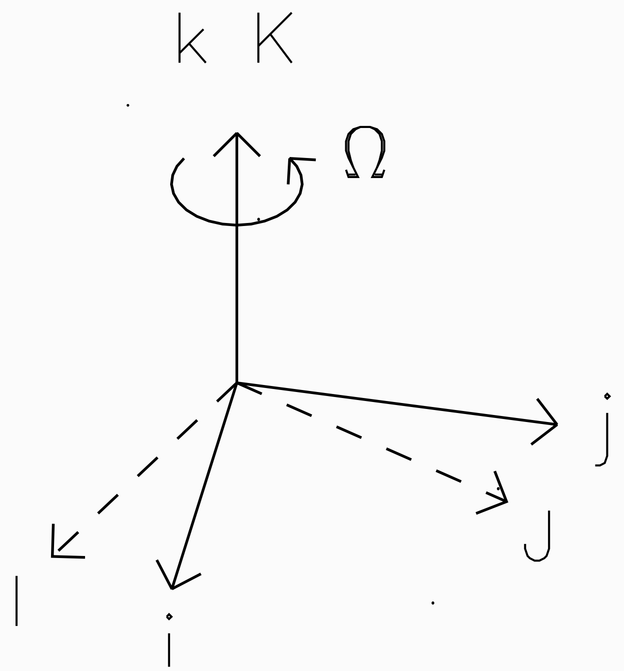
- Acoustic waves
- Inertio-gravity waves
- Phase velocity and group velocity

- Some key conservation properties

The multiscale nature of atmospheric dynamics



Dynamics in a rotating frame



Let

$$\mathbf{A} = A_x\mathbf{I} + A_y\mathbf{J} + A_z\mathbf{K}$$

$$= a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

Time rate of change of an arbitrary vector

$$\begin{aligned}
\left(\frac{D\mathbf{A}}{Dt}\right)_{\mathrm{IF}} &= \mathbf{I}\frac{DA_x}{Dt} + \mathbf{J}\frac{DA_y}{Dt} + \mathbf{K}\frac{DA_z}{Dt} \\
&= \mathbf{i}\frac{Da_x}{Dt} + \mathbf{j}\frac{Da_y}{Dt} + \mathbf{k}\frac{Da_z}{Dt} \\
&+ a_x\left(\frac{D\mathbf{i}}{Dt}\right)_{\mathrm{IF}} + a_y\left(\frac{D\mathbf{j}}{Dt}\right)_{\mathrm{IF}} + a_z\left(\frac{D\mathbf{k}}{Dt}\right)_{\mathrm{IF}} \\
&= \left(\frac{D\mathbf{A}}{Dt}\right)_{\mathrm{RF}} + \mathbf{\Omega}\times\mathbf{A}
\end{aligned}$$

Apply to position vector

$$\mathbf{u}_{\mathrm{IF}} = \mathbf{u}_{\mathrm{RF}} + \boldsymbol{\Omega} \times \mathbf{x}$$

Apply to velocity vector

$$\mathbf{a}_{\mathrm{IF}} = \mathbf{a}_{\mathrm{RF}} + 2\boldsymbol{\Omega} \times \mathbf{u}_{\mathrm{RF}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x})$$

Governing equations

Mass

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0$$

Thermodynamics

$$\frac{D\theta}{Dt} = Q$$

Momentum

$$\frac{D\mathbf{u}}{Dt} + 2\mathbf{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla\Phi + \mathbf{F}$$

along with

$$p = RT\rho$$

and

$$T = \left(\frac{p}{p_{00}}\right)^{\kappa} \theta = \Pi(p)\theta$$

where $\kappa = R/C_p$.

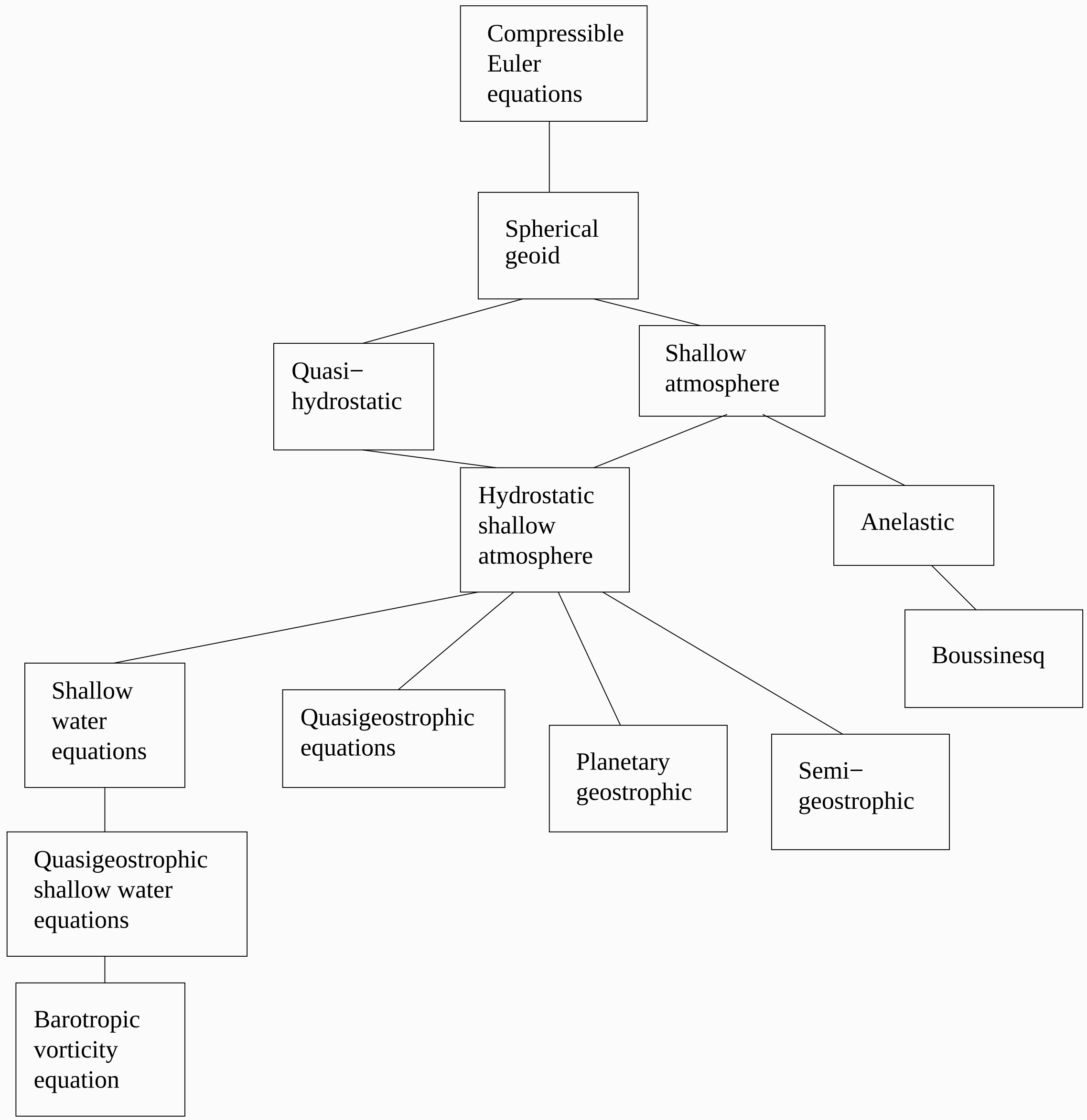
Some common approximations

- Spherical geoid: $\Phi = \Phi(r)$
- Quasi-hydrostatic: neglect Dw/Dt
- Anelastic: $\nabla(\rho_0 \mathbf{u}) = 0$
- Shallow atmosphere:

  neglect Coriolis terms involving horizontal component of $\mathbf{\Omega}$;

  replace $1/r$ by $1/a$;

  neglect some metric terms.


Compressible Euler equations
Spherical geoid
Quasi–hydrostatic
Shallow atmosphere
Hydrostatic shallow atmosphere
Anelastic
Boussinesq
Shallow water equations
Quasigeostrophic equations
Planetary geostrophic
Semi–geostrophic
Quasigeostrophic shallow water equations
Barotropic vorticity equation

The importance of waves

- Acoustic waves: very fast, energetically very weak
- Inertio-gravity waves: fast, energetically weak
- Rossby waves and balanced vortical motion: energetically dominant

But fast waves are crucial for **adjustment** towards balance.

Acoustic waves

Neglect Coriolis and gravity

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0$$

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p$$

Linearize about an isothermal state of rest

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla.(\mathbf{u}) = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_0}\nabla p = -\frac{c^2}{\rho_0}\nabla \rho$$

where $c^2 = \partial p / \partial \rho|_\theta = RT_0/(1-\kappa)$

$$\frac{\partial^2 \rho}{\partial t^2} - c^2 \nabla^2 \rho = 0$$

Seek wavelike solutions $\propto \exp\{i\mathbf{k}.\mathbf{x} - \omega t\}$

to obtain the **dispersion relation**

$$\omega^2 = c^2|\mathbf{k}|^2$$

Note $\mathbf{u}||\mathbf{k}$: waves are **longitudinal**

Also, acoustic waves are **non-dispersive**

Inertio-gravity waves

Make the Boussinesq approximation: assume the fluid to be incompresible $\nabla.\mathbf{u} = 0$, and neglect variations in density except where they appear in a buoyancy term, i.e. multiplied by g.

Also neglect Coriolis terms involving the horizontal component of $\mathbf{\Omega}$.

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x},$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y},$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + b,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

$$\frac{Db}{Dt} + wN^2 = 0,$$

where

$$b = -g\frac{\rho'}{\rho_0} \qquad \text{and} \qquad N^2 = -\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z}$$

Linearize about a hydrostatic state of rest

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x},$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y},$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + b,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

$$\frac{\partial b}{\partial t} + wN^2 = 0.$$

Seek wavelike solutions $\propto e^{\mathrm{i}(kx+ly+mz-\omega t)}$ to obtain the **dispersion relation**

$$\omega^2 = \frac{(k^2+l^2)N^2 + m^2 f^2}{k^2+l^2+m^2}$$

Note $\mathbf{u} \perp \mathbf{k}$: waves are **transverse**

For very deep waves, $m^2/(k^2+l^2) \ll 1$, $\omega^2 \approx N^2$

For very shallow waves, $(k^2+l^2)/m^2 \ll 1$, $\omega^2 \approx f^2$

Phase velocity

If a wavelike disturbance is $\propto \exp\{\mathrm{i}\phi(\mathbf{x}, t)\}$

wave crests and troughs are surfaces of constant phase ϕ.

For a plane wave

$$\begin{aligned} \phi &= \mathbf{k}.\mathbf{x} - \omega t \\ &= \mathbf{k}.(\mathbf{x} - \mathbf{c}_p t) \end{aligned}$$

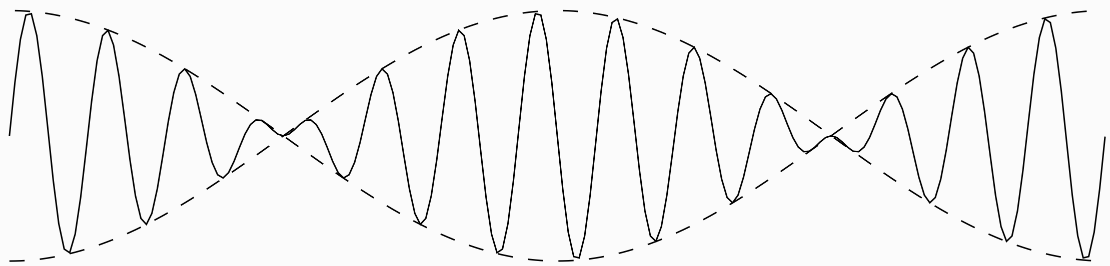
where $\mathbf{c}_p = \mathbf{k}\omega/|\mathbf{k}|^2$ is the **phase velocity**. Crests and troughs move at velocity $\mathbf{c}_p$.

Group velocity

How does a **packet** of waves propagate?

Consider a superposition of two 1D waves with similar wavenumber and frequency, satisfying $\omega = \omega(k)$

$$\begin{aligned} q &= \frac{1}{2}\left(e^{\mathrm{i}\{(k+\delta k)x-(\omega+\delta\omega)t\}} + e^{\mathrm{i}\{(k-\delta k)x-(\omega-\delta\omega)t\}}\right) \\ &= \cos(\delta k\, x - \delta\omega\, t)e^{\mathrm{i}\{kx-\omega t\}} \end{aligned} \tag{1}$$



Individual crests and troughs propagate at the **phase velocity**

$$c_p = \omega/k$$

but wave packets propagate at the **group velocity**

$$c_g = \delta\omega/\delta k \to \partial\omega/\partial k$$

In 3D

$$\mathbf{c}_g = \nabla_{\mathbf{k}}\omega = \left(\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}, \frac{\partial\omega}{\partial m}\right)$$

Matlab demo of phase and group velocity

Some key conservation properties

Some conservation properties can be expressed as

$$\frac{\partial \mathcal{A}}{\partial t} + \nabla.(\mathbf{F}) = 0$$

Mass

$$\mathcal{A} = \rho \qquad \text{and} \qquad \mathbf{F} = \rho \mathbf{u}$$

Angular momentum

$$\mathcal{A} = \rho \widehat{\mathbf{z}}. \left[\mathbf{r} \times (\mathbf{u} + \boldsymbol{\Omega} \times \mathbf{r}) \right] \qquad \text{and} \qquad \mathbf{F} = \mathbf{u}\mathcal{A} + p\widehat{\mathbf{z}} \times \mathbf{r}$$

Energy

$$\mathcal{A} = \rho(\mathbf{u}^2/2 + C_v T + \Phi) \qquad \text{and} \qquad \mathbf{F} = \mathbf{u}(\mathcal{A} + p)$$

Potential temperature

$$\frac{D\theta}{Dt} = 0$$

Potential vorticity

$$\frac{DQ}{Dt} = 0$$

where $Q = \boldsymbol{\zeta}.\nabla\theta/\rho$

Potential enstrophy

$$\mathcal{A} = \rho Q^2/2 \qquad \text{and} \qquad \mathbf{F} = \mathbf{u}\mathcal{A}$$