Basic Dynamics - II

Tuesday 3 June, 2008

Outline

- Hydrostatic balance
- Geostrophic balance
- Quasigeostrophic theory; Rossby waves
- Eulerian and Lagrangian timescales
- Turbulence; energy and potential enstrophy cascades

Hydrostatic balance

Scale analysis of vertical momentum equation (following Holton)

w-equation	$\frac{Dw}{Dt}$	$-\frac{u^2+v^2}{r}$	$-2\Omega u \cos\phi$	g	$\frac{1}{\rho}\frac{\partial p}{\partial r}$
Scales	UW/L	U^2/a	$f_0 U$	g	$P_0/\rho H$
Values ms^{-2}	10^{-7}	10^{-5}	10^{-3}	10	10

Clearly

$$g + \frac{1}{\rho}\frac{\partial p}{\partial r} \approx 0$$

Geostrophic balance

Scale analysis of horizontal momentum equation (following Holton)

u-equation	$\frac{Du}{Dt}$	$-\frac{uv\tan\phi}{r}$	$\frac{uw}{r}$	$-2\Omega v\sin\phi$	$2\Omega w\cos\phi$	$\frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial\lambda}$
v-equation	$\frac{Dv}{Dt}$	$-\frac{u^2\tan\phi}{r}$	$\frac{vw}{r}$	$2\Omega u\sin\phi$		$\frac{1}{\rho r}\frac{\partial p}{\partial\phi}$
Scales	U^2/L	U^2/a	UW/a	$f_0 U$	$f_0 W$	$\delta P/\rho L$
Values ms^{-2}	10^{-4}	10^{-5}	10^{-8}	10^{-3}	10^{-6}	10^{-3}

Then

$$u \approx -\frac{1}{f_0\rho r}\frac{\partial p}{\partial\phi} \equiv u_g; \qquad v \approx \frac{1}{f_0\rho r\cos\phi}\frac{\partial p}{\partial\lambda} \equiv v_g$$

$$\text{Rossby number :} \qquad Ro = U/(f_0 L)$$

Conditions for validity of hydrostatic approximation

Can neglect Dw/Dt when

$$\frac{UW}{L} \ll \frac{\delta P}{\rho H}$$

From horizontal momentum equation

$$\frac{\delta P}{\rho} \sim U^2 \quad \text{or} \quad f_0 L U$$

So we require

$$\frac{WH}{UL} \ll 1 \quad \text{or} \quad \frac{WH}{UL} Ro \ll 1$$

From the mass continuity equation

$$\frac{W}{U} \sim \frac{H}{L} \quad \text{or} \quad \frac{W}{U} \sim \frac{H}{L} Ro$$

so hydrostatic balance will be a good approximation provided

$$\frac{H^2}{L^2} \ll 1 \quad \text{or} \quad \frac{H^2}{L^2} Ro^2 \ll 1$$

Alternatively

$$\omega^2 \ll N^2$$

Hydrostatic **balance** corresponds to the **absence of certain kinds of waves**, namely internal acoustic and fast inertio-gravity waves.

Mathematically, it leads to a **nonlocal** 1D boundary value problem for w.

It can be thought of as an asymptotic limit in which some information propagates infinitely fast, so part of the **adjustment** process is **instantaneous**.

Numerical methods should be able to capture such important asymptotic limits.

Sketch of quasigeostrophic theory

Work in β-plane geometry $f = f_0 + \beta y$ and a log-pressure vertical coordinate $\tilde{z}$. Assume hydrostatic balance, that $Ro \ll 1$, that thermodynamic quantities are close to reference profiles, and that $\beta L / f_0 \ll 1$.

At leading order $u \approx u_g$, $v \approx v_g$.

$$u_g = -\frac{\partial \psi}{\partial y}; \qquad v_g = \frac{\partial \psi}{\partial x}; \qquad \frac{\theta'}{\theta_{\mathrm{ref}}} = \frac{f}{g}\frac{\partial \psi}{\partial \tilde{z}}$$

where $\psi = \Phi' / f_0$.

So let $u = u_g + u_a$, $v = v_g + v_a$

At next order we get a vorticity equation

$$\frac{D_g \zeta_g}{Dt} = \frac{f_0}{\rho_0} \frac{\partial}{\partial \tilde{z}} \left(\rho_0 \tilde{w} \right)$$

which is combined with the thermodynamic equation

$$\frac{D_g \theta'}{Dt} + \tilde{w} \frac{\partial \theta_0}{\partial \tilde{z}} = 0$$

to form the potential vorticity equation

$$\frac{D_g q}{Dt} = 0$$

where

$$q = f_0 + \beta y + \nabla_{\tilde{z}}^2 \psi + \frac{1}{\rho_0} \frac{\partial}{\partial \tilde{z}} \left(\rho_0 \frac{f_0^2}{N_{\mathrm{ref}}^2} \frac{\partial \psi}{\partial \tilde{z}} \right)$$

Quasigeostrophic theory filters out all fast waves

Mathematically, it leads to a **nonlocal** 3D boundary value problem for ψ.

It can be thought of as an asymptotic limit in which some information propagates infinitely fast, so the **adjustment** process is **instantaneous**.

Again, numerical methods should be able to capture such important asymptotic limits.

Quasigeostrophic theory embodies **advection**, i.e. material conservation, and **invertibility** of potential vorticity.

These two properties can be used to understand many GFD phenomena

E.g. Rossby waves

Linearize PV advection and invertibility equations about a state of rest:

$$\frac{\partial q}{\partial t} + \beta v_g = 0$$

$$q = \nabla_{\tilde{z}}^2 \psi + \frac{1}{\rho_0} \frac{\partial}{\partial \tilde{z}} \left(\rho_0 \frac{f_0^2}{N_{\mathrm{ref}}^2} \frac{\partial \psi}{\partial \tilde{z}} \right)$$

Seek wavelike solutions

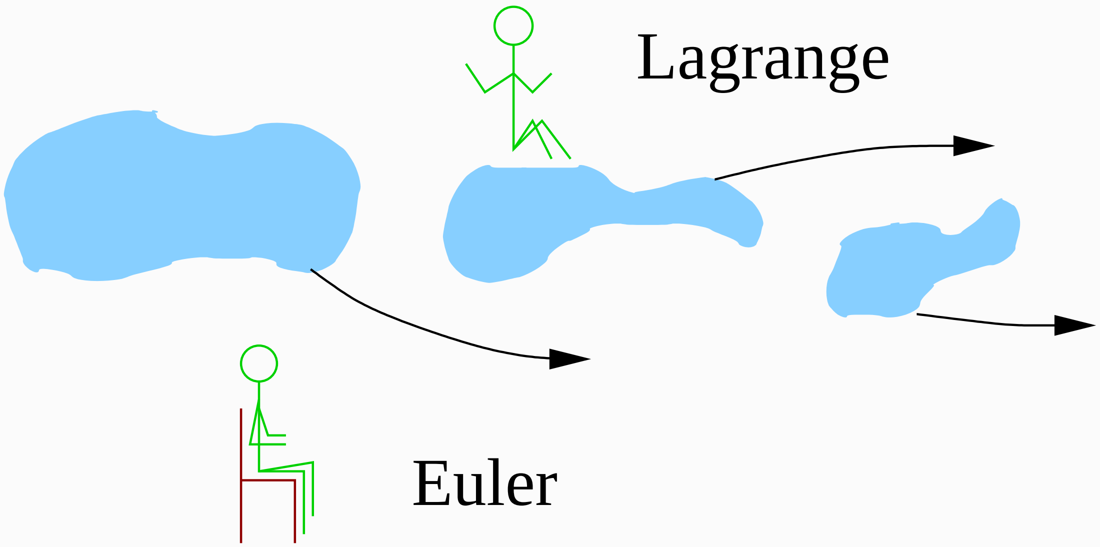
$$\psi = Re \left\{ \hat{\psi}(\tilde{z}) \exp \mathrm{i}(kx + ly + m\tilde{z} - \omega t) \right\}$$

to obtain

$$\omega = - \frac{\beta k}{k^2 + l^2 + (m^2 + 1/4H_\rho^2) f_0^2 / N_{\mathrm{ref}}^2}$$

Matlab demo of Rossby wave

Eulerian and Lagrangian timescales



Large scale atmospheric flow has a steep energy spectrum, something like k^{-3}

Eulerian timescale

$$\tau_{\mathrm{Eul}} \sim \frac{L}{U} \sim \frac{1}{kU_0}$$

Lagrangian timescale

Some quantities are approximately materially conserved $D\chi/Dt \approx 0$ and so have long Lagrangian timescale.

For other quantities

$$\tau_{\mathrm{Lag}} \sim \frac{1}{S} \sim \frac{1}{k_0 U_0}$$

This fact can be exploited by approximating time derivatives in a Lagrangian way.

However...

an important exception is flow over orography, for which τ_{Eul} becomes very long, but

$$\tau_{\mathrm{Lag}} \sim \frac{L}{U} \sim \frac{1}{kU_0}$$

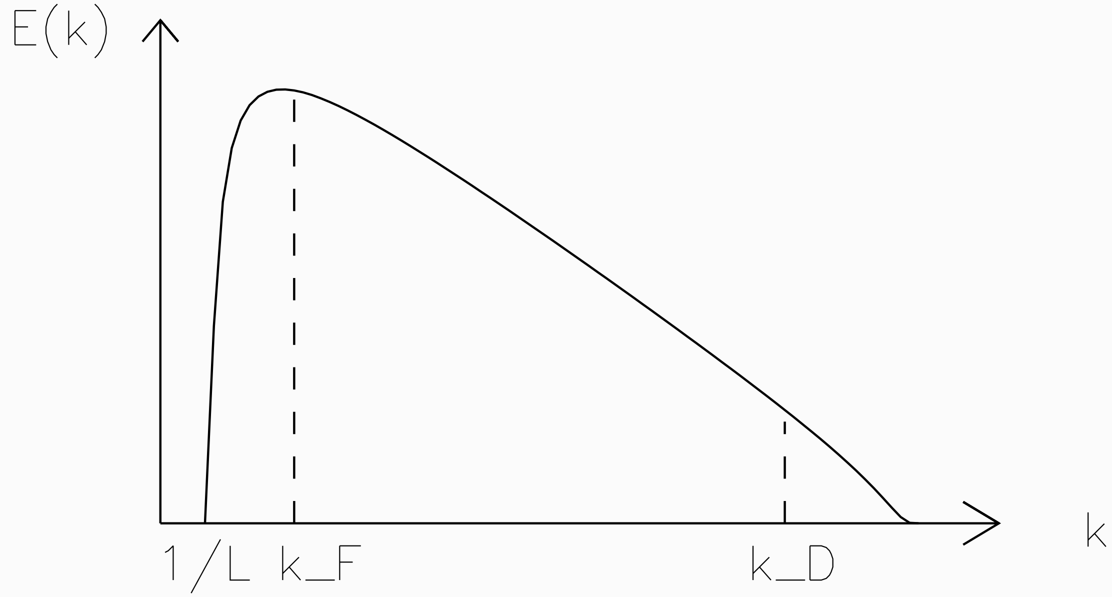
In this case (semi-)Lagrangian methods with long time steps suffer from spurious **orographic resonance**.

Turbulence and cascades

Nonlinearity implies **interaction of scales**.

Dynamics attempts to generate variability at and below the grid scale.

Kolmogorov (1941) theory



For 3D, statistically steady, homogeneous, isotropic turbulence, in an **inertial range**:

At wavenumber k, the only dimensional quantities are the energy throughput ε and k itself.

$$[E(k)] = L^3 T^{-2}$$

$$[\varepsilon] = L^2 T^{-3} \quad \text{and} \quad [k] = L^{-1}$$

so

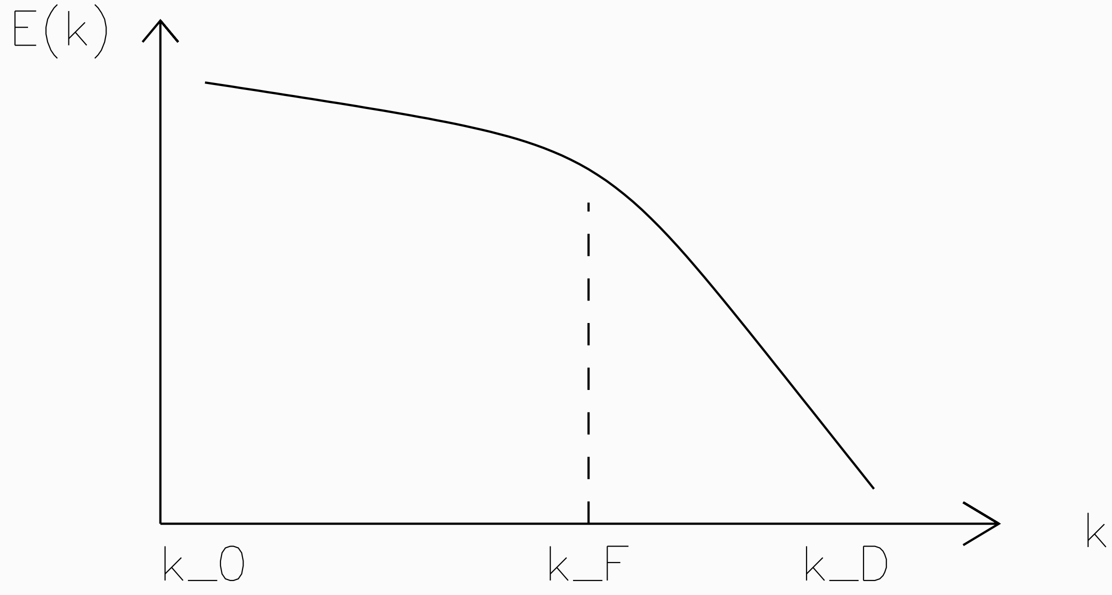
$$E(k) = C_1 \varepsilon^{2/3} k^{-5/3}$$

for some universal C_1 of order 1.

Two dimensional turbulence

In 2D turbulence we have another conservable quantity, the enstrophy, and therefore a cascade of enstrophy η.

Typically energy now cascades upscale while enstrophy cascades downscale.



In the energy cascade region $E(k) = C\varepsilon^{2/3}k^{-5/3}$, as before.

In the enstrophy cascade region

$$[E(k)] = L^3T^{-2}$$

$$[\eta] = T^{-3} \quad \text{and} \quad [k] = L^{-1}$$

so

$$E(k) = C_2\eta^{2/3}k^{-3}$$

for some universal C_2 of order 1.

Energy upscale, enstrophy downscale (mostly)

Let

$$E = \int E(k)\,dk \quad \text{and} \quad Z = \int k^2 E(k)\,dk$$

Suppose energy is initially concentrated near wavenumber k_1 and subsequently spreads out, so that

$$\frac{d}{dt}\int (k - k_1)^2 E(k)\,dk > 0$$

The fact that E and Z are conserved (neglecting viscosity) implies

$$\frac{d}{dt}\left(\frac{\int kE(k)\,dk}{\int E(k)\,dk}\right) < 0$$

Similarly, assuming

$$\frac{d}{dt}\int (k^2 - k_1^2)^2 E(k)\, dk > 0$$

implies

$$\frac{d}{dt}\left(\frac{\int k^2 Z(k)\, dk}{\int Z(k)\, dk}\right) > 0$$

Matlab demo of cascades