# Horizontal discretizations

Wednesday 4 June, 2008

## Outline

- Wave propagation and staggered grids

  1D gravity waves

  2D inertio-gravity waves

  2D Rossby waves

- Conservation properties

(I will not discuss the spectral method, nor advection schemes)

## Wave propagation and staggered grids

Motivation: accurate representation of propagation of **fast waves** and hence **adjustment** and **balance**.

## 1D gravity waves

Linearized, 1D, non-rotating, shallow water equations

$$\frac{\partial \Phi}{\partial t} + \Phi_0 \frac{\partial u}{\partial x} = 0 \qquad \frac{\partial u}{\partial t} + \frac{\partial \Phi}{\partial x} = 0$$

Look for wavelike solutions

$$\Phi = Re\left\{\hat{\Phi}\exp[\mathrm{i}(kx - \omega t)]\right\} \qquad u = Re\left\{\hat{u}\exp[\mathrm{i}(kx - \omega t)]\right\}$$

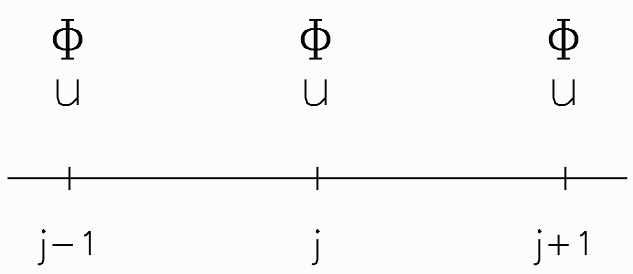
to find the dispersion relation

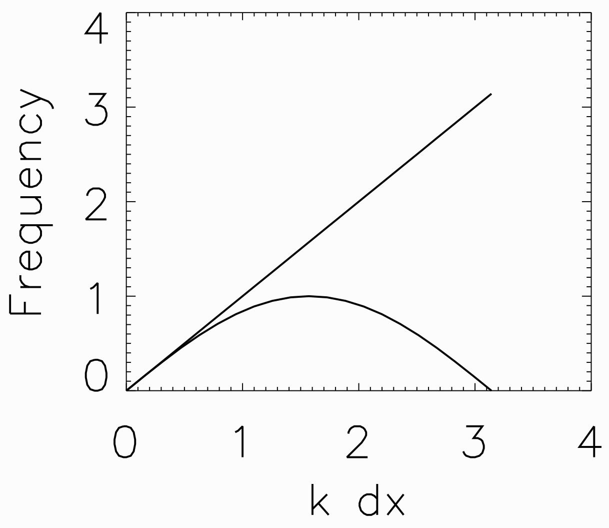
$$\omega^2 = k^2 \Phi_0$$

# On an unstaggered grid

$$\frac{\partial u_j}{\partial t} + \frac{\Phi_{j+1} - \Phi_{j-1}}{2\Delta x} = 0 \qquad \frac{\partial \Phi_j}{\partial t} + \frac{u_{j+1} - u_{j-1}}{2\Delta x} = 0$$

$$k \rightarrow \sin(k\Delta x)/\Delta x$$

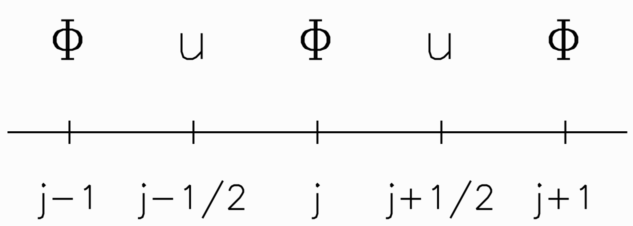


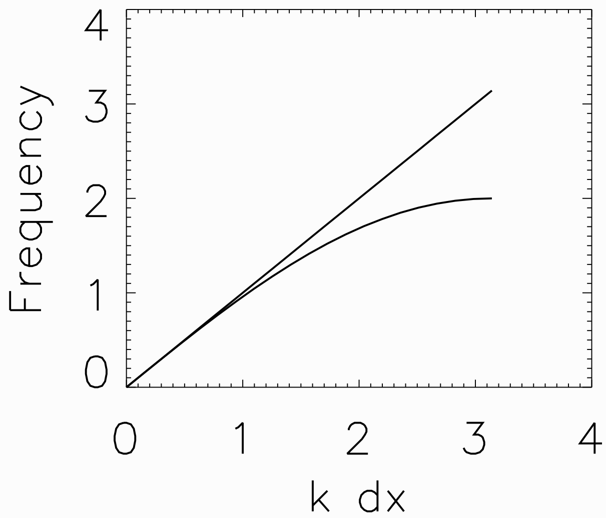



# On a staggered grid

$$\frac{\partial u_{j+1/2}}{\partial t} + \frac{\Phi_{j+1} - \Phi_j}{\Delta x} = 0 \qquad \frac{\partial \Phi_j}{\partial t} + \frac{u_{j+1/2} - u_{j-1/2}}{\Delta x} = 0$$

$$k \rightarrow \sin(k\Delta x/2)/(\Delta x/2)$$



## Matlab demo of unstaggered and staggered grids

## Computational modes

Spuriously fail to propagate

## Parasitic modes

Spurious propagation, e.g. wrong sign of group velocity

Implications for inhomogeneous and adaptive grids

**Matlab demo of parasitic mode**

## 2D inertio-gravity waves

Linearized, 2D, rotating, shallow water equations

$$
\begin{aligned}
\frac{\partial \Phi}{\partial t} + \Phi_0 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0 \\
\frac{\partial u}{\partial t} - fv + \frac{\partial \Phi}{\partial x} &= 0 \\
\frac{\partial v}{\partial t} + fu + \frac{\partial \Phi}{\partial y} &= 0
\end{aligned}
$$

Dispersion relation (for $f = f_0 = \mathrm{const}$)

$$
\omega \left( \omega^2 - f_0^2 - (k^2 + l^2)\Phi_0 \right) = 0
$$

## The Rossby radius

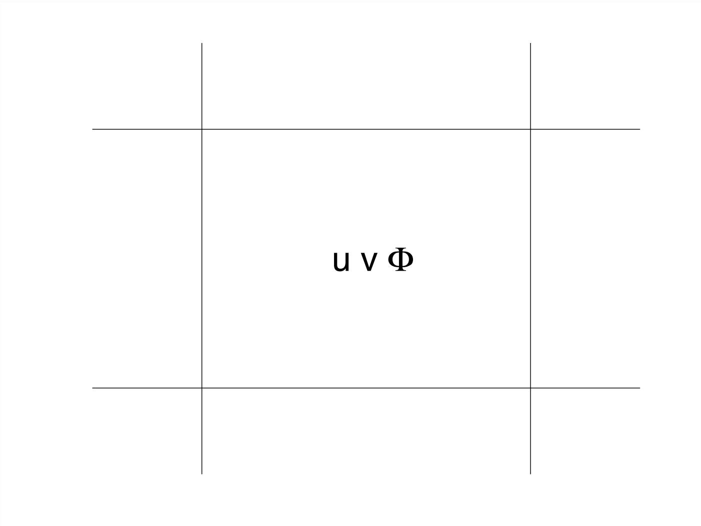
Define

$$\lambda = \Phi_0^{1/2} / f_0$$

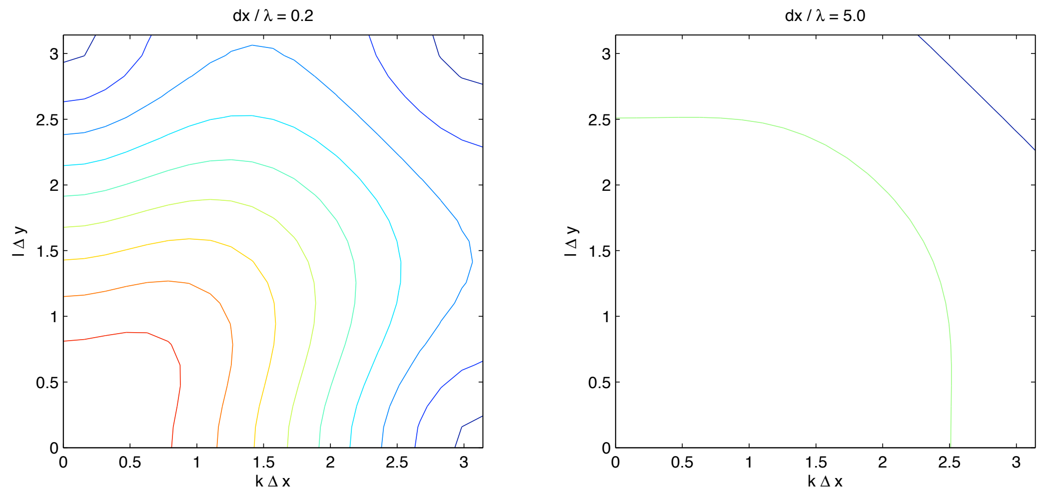
On scales smaller than $\lambda$ pressure gradient terms dominate.

On scales greater than $\lambda$ Coriolis terms dominate.

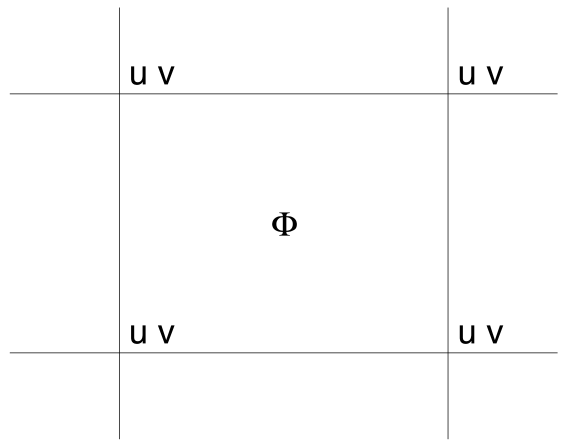
# A-grid



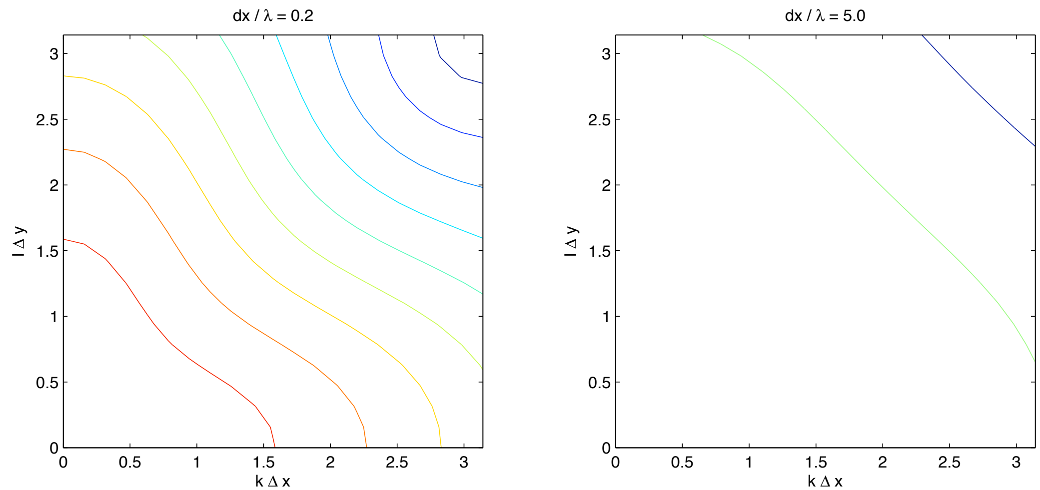

$$k \longrightarrow \sin(k\Delta x)/\Delta x$$
$$l \longrightarrow \sin(l\Delta y)/\Delta y$$
$$f_0 \longrightarrow f_0$$



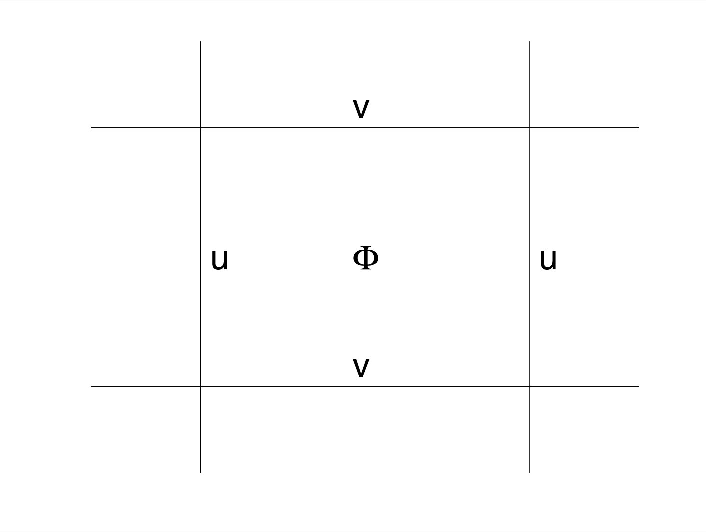
# B-grid



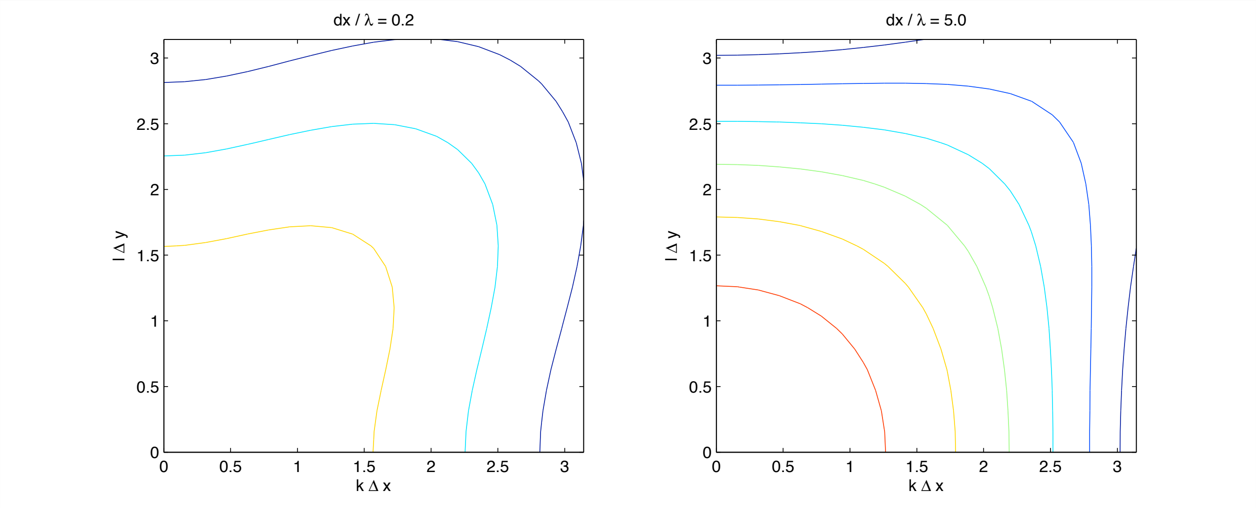

$$k \longrightarrow \cos(l\Delta y/2)\sin(k\Delta x/2)/(\Delta x/2)$$
$$l \longrightarrow \cos(k\Delta x/2)\sin(l\Delta y/2)/(\Delta y/2)$$
$$f_0 \longrightarrow f_0$$



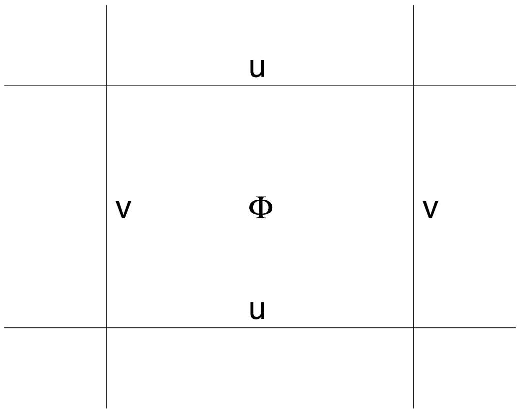
# C-grid



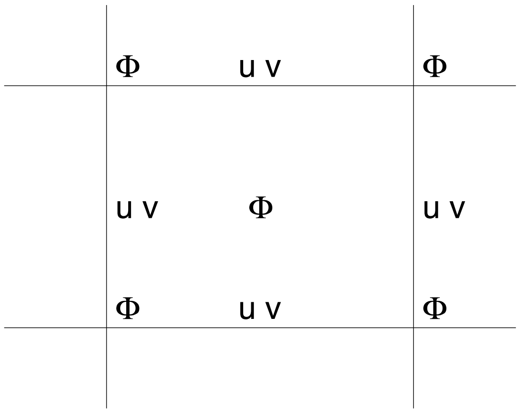

$$k \rightarrow \sin(k\Delta x/2)/(\Delta x/2)$$
$$l \rightarrow \sin(l\Delta y/2)/(\Delta y/2)$$
$$f_0 \rightarrow f_0 \cos(k\Delta x/2)\cos(l\Delta y/2)$$



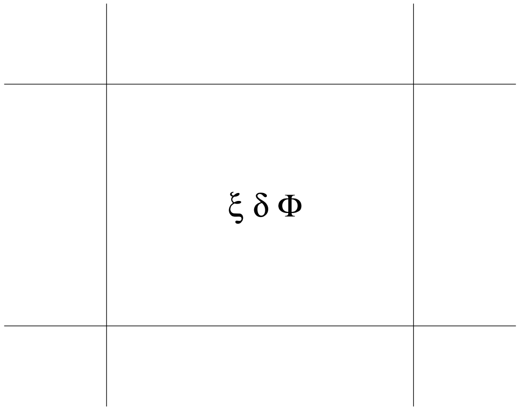
## D-grid




## E-grid




## Z-grid



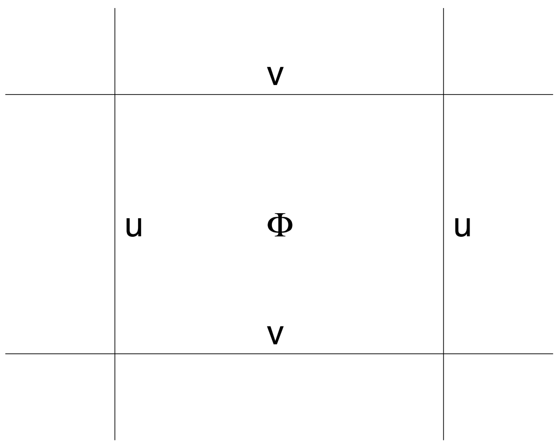
**Matlab demo of geostrophic adjustment**

There exist analogues of these staggered grids for other grid cell shapes such as triangles and hexagons - see the lecture by Todd Ringler.

Also, analogous ideas hold for finite-element discretizations.

## Rossby wave propagation on the C-grid

Does averaging of Coriolis terms lead to poor representation of Rossby wave propagation?




When $f$ is a function of position there are various ways of averaging the Coriolis terms

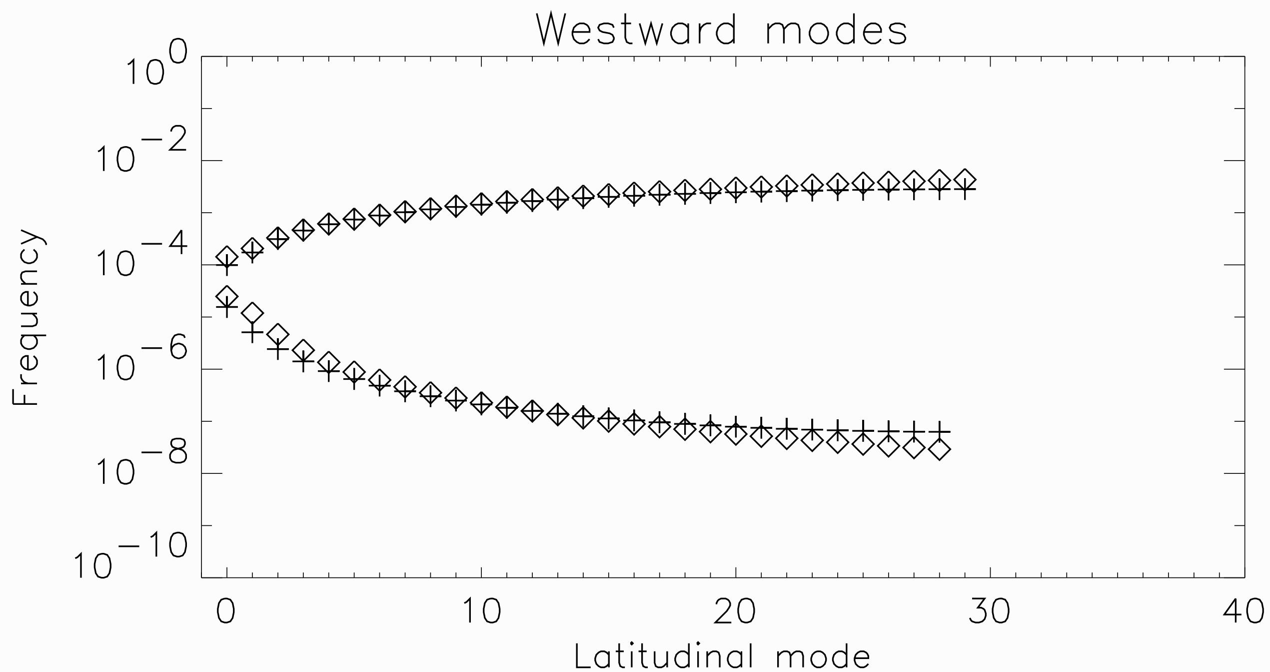
$f$-at-$\Phi$-points is quite good:

$$\partial_t u - \overline{f\overline{v}^y}^x + \delta_x \Phi = 0$$

$$\partial_t v + \overline{f\overline{u}^x}^y + \delta_y \Phi = 0$$

It captures the wave frequency quite well even for short north-south wavelengths (but not short east-west wavelengths).

It is also energy conserving.




Shallow water normal modes M= 2

Omega=7.29200e−05 Phi_0=100000.

Variables: wind cpts

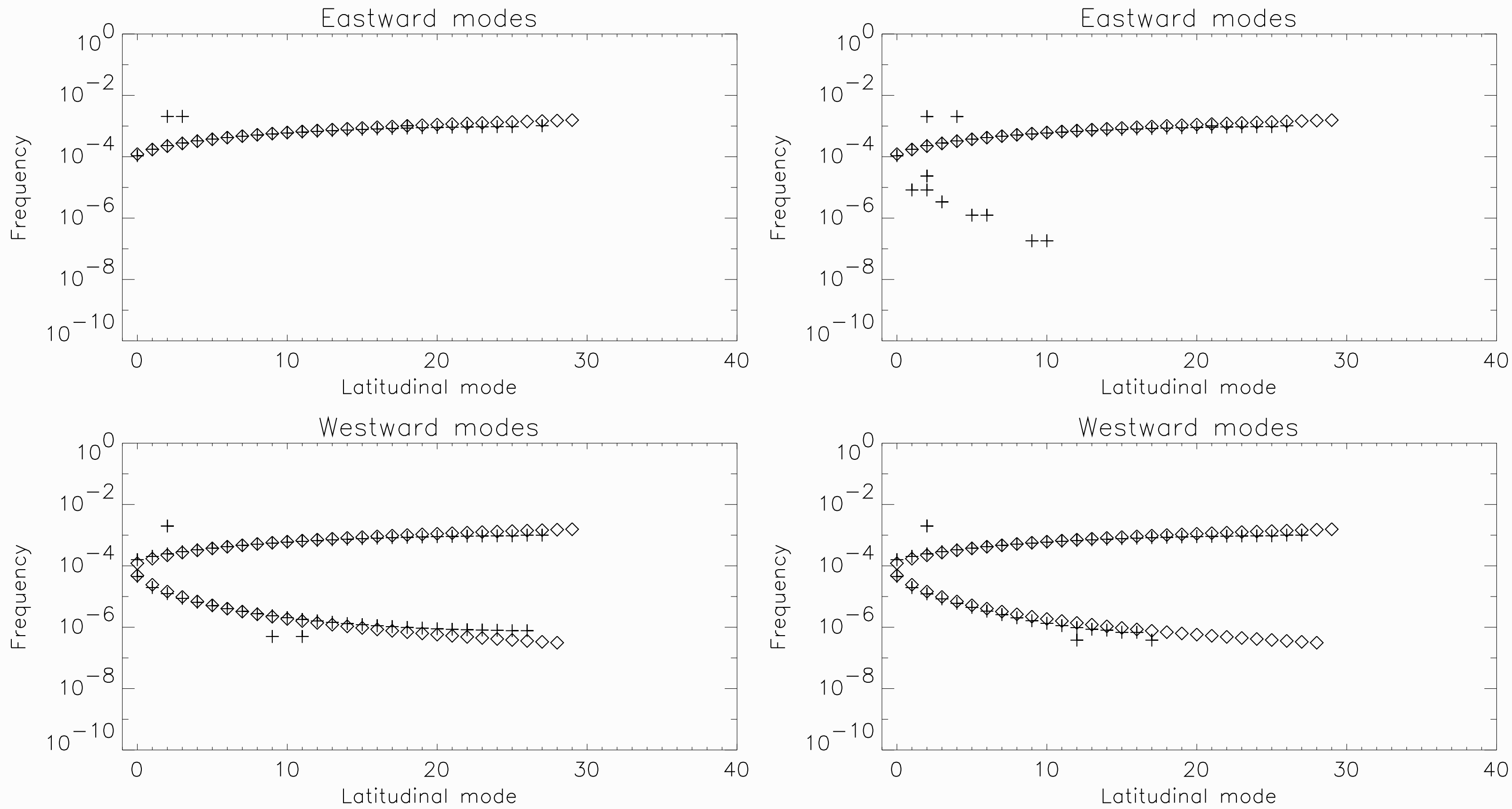
Beta−plane channel NLAT = 30

Lat. coord.: phi

**In spherical geometry** it is important to include appropriate geometrical factors in the averaging of the Coriolis terms, for consistency with the mass continuity equation:

$$\frac{\partial u}{\partial t} - \overline{\frac{f}{\cos\phi}\overline{v\cos\phi}^{\phi}}^{\lambda} + \frac{1}{a\cos\phi}\delta_\lambda \Phi = 0$$

$$\frac{\partial v}{\partial t} + \overline{f\overline{u}^{\lambda}}^{\phi} + \frac{1}{a}\delta_\phi \Phi = 0$$


Eastward modes
Eastward modes
Westward modes
Westward modes
Frequency
Latitudinal mode
$10^{0}$
$10^{-2}$
$10^{-4}$
$10^{-6}$
$10^{-8}$
$10^{-10}$
0
10
20
30
40

## Energy conservation - Coriolis terms

Coriolis terms should cancel when we take $u$ times

$$\frac{Du}{Dt} - fv = \ldots$$

plus $v$ times

$$\frac{Dv}{Dt} + fu = \ldots$$

Straightforward for A-grid and B-grid

For the C-grid, write $u^* = u\Phi a\Delta\phi$, $v^* = v\Phi a\cos\phi\Delta\lambda$

$$\frac{\partial}{\partial t}(ua\cos\phi\Delta\lambda)_{i,j+1/2}$$
$$-\alpha_{i,j+1/2}\, v^*_{i+1/2,j+1} - \beta_{i,j+1/2}\, v^*_{i-1/2,j+1}$$
$$-\gamma_{i,j+1/2}\, v^*_{i-1/2,j} - \delta_{i,j+1/2}\, v^*_{i+1/2,j} = \dots$$

$$\frac{\partial}{\partial t}(va\Delta\phi)_{i+1/2,j}$$
$$+\alpha_{i,j-1/2}\, u^*_{i,j-1/2} + \beta_{i+1,j-1/2}\, u^*_{i+1,j-1/2}$$
$$+\gamma_{i+1,j+1/2}\, u^*_{i+1,j+1/2} + \delta_{i,j+1/2}\, u^*_{i,j+1/2} = \dots$$

## Energy conservation - pressure gradient terms

We require a discrete analogue of

$$\mathbf{v}.\nabla p + p\nabla.\mathbf{v} = \nabla.(\mathbf{v}p)$$

or, at least,

$$\int \mathbf{v}.\nabla p \, dA + \int p\nabla.\mathbf{v} \, dA = 0$$

Relatively straightforward on the A-grid.

E.g. on a C-grid write $\hat{u} = ua\Delta\phi = u^*/\rho$, $\hat{v} = va\cos\phi\Delta\lambda = v^*/\rho$

Then

$$\begin{aligned}
&\Sigma_{i\,j}\,\hat{u}_{i,j+1/2}\,\left(p_{i+1/2,j+1/2} - p_{i-1/2,j+1/2}\right) + \\
&\Sigma_{i\,j}\,\hat{v}_{i+1/2,j}\,\left(p_{i+1/2,j+1/2} - p_{i+1/2,j-1/2}\right) + \\
&\Sigma_{i\,j}\,p_{i+1/2,j+1/2}\,\left(\hat{u}_{i+1,j+1/2} - \hat{u}_{i,j+1/2}\right) + \\
&\Sigma_{i\,j}\,p_{i+1/2,j+1/2}\,\left(\hat{v}_{i+1/2,j+1} - \hat{v}_{i+1/2,j}\right) = 0
\end{aligned}$$

(with care at the poles)

Similar considerations apply to **angular momentum conservation**.

**conservation of potential enstrophy** is also possible, at least in the shallow water case, but is more complicated.